Linear Equations

A linear first-order DE looks like

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Standard form is

$$\frac{dy}{dx} + P(x)y = Q(x)$$

To solve first-order linear DEs, we will be using the term $e^{\int P(x)dx}$, called the <u>integrating factor</u>.

Solving first-order linear DE

1) Put into standard form $\frac{dy}{dx}+P(x)y=Q(x)$

2) Find the integrating factor $e^{\int P(x)dx}$

3) Multiply both sides of the DE by the integrating factor. The result will be:

$$\frac{d}{dx}\left[e^{\int P(x)dx}y\right]=Q(x)e^{\int P(x)dx}$$

4) Integrate both sides of the equation.

Ex. Solve $\frac{dy}{dx} \underbrace{-3}_{P(x)} y = 0$

$e^{\int -3\,dx} = e^{-3x}$

$e^{-3x}\left[\frac{dy}{dx} - 3y = 0\right]$

$\int \frac{d}{dx}\left[e^{-3x} \cdot y\right] = \int 0$

$e^{-3x} \cdot y = C$

$\boxed{y = Ce^{3x}}$

$\underline{\text{Ex.}}$ Solve $\left(\frac{dy}{dx} - 3y = 6\right) e^{-3x}$

$e^{\int -3\,dx} = e^{-3x}$

$\frac{d}{dx}\left[e^{-3x} \cdot y\right] = 6e^{-3x}$

$e^{-3x} \cdot y = -2e^{-3x} + C$

$$\boxed{y = -2 + Ce^{3x}}$$

$y = \underbrace{x+5} + \underbrace{Ce^{-2x}}_{\to 0}$

<u>Transient terms</u> are terms that approach zero as x goes to infinity.

Ex. Solve $\frac{x}{x}\frac{dy}{dx} - \frac{4y}{x} = \frac{x^6 e^x}{x}$

$$e^{\int -\frac{4}{x}dx} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}$$

$$\left(\frac{dy}{dx} - \frac{4}{x}y = x^5 e^x\right)x^{-4}$$

$$\int \frac{d}{dx}\left[x^{-4}\cdot y\right] = \int x e^x$$

$$x^{-4}y = xe^x - e^x + C$$

$$y = x^5 e^x - x^4 e^x + Cx^4$$

$$(0, \infty)$$

What is the interval of definition?

Ex. Solve $\left(x^2-9\right)\frac{dy}{dx}+xy=0$

$$\frac{dy}{dx}+\frac{x}{x^2-9}y=0$$

$$e^{\int \frac{x}{x^2-9}dx}=e^{\frac{1}{2}\ln(x^2-9)}$$

$$=e^{\ln(x^2-9)^{1/2}}=\sqrt{x^2-9}$$

$$\frac{d}{dx}\left[\sqrt{x^2-9}\cdot y\right]=0$$

$$\sqrt{x^2-9}\;y=C$$

$$y=\frac{C}{\sqrt{x^2-9}}$$

Ex. Solve $\frac{dy}{dx} + y = x,\ y(0) = 4$

$$\boxed{e^{\int 1\,dx} = e^x}$$

$$\frac{d}{dx}\left[e^x \cdot y\right] = xe^x$$

$$e^x \cdot y = xe^x - e^x + C$$

$$y = x - 1 + Ce^{-x}$$

$$4 = 0 - 1 + Ce^0$$

$$C = 5$$

$$y = x - 1 + 5e^{-x}$$

<u>Ex.</u> Solve $\frac{dy}{dx} - 2xy = 2,\ y(0) = 1$

$$e^{\int -2x\,dx} = e^{-x^2}$$

$$\frac{d}{dx}\left[e^{-x^2} \cdot y\right] = 2e^{-x^2}$$

$$e^{-x^2} \cdot y = \int_0^x 2e^{-t^2}\,dt + C$$

$$y = e^{x^2}\int_0^x 2e^{-t^2}\,dt + Ce^{x^2}$$

$$1 = e^0 \int_0^0 \text{---}\,dt + Ce^0 \longrightarrow C = 1$$

$$y = e^{x^2}\int_0^x e^{-t^2}\,dt + e^{x^2}$$

<u>Ex.</u> Solve $\frac{dy}{dx} + y = f(x)$, $y(0) = 0$, where

$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

$\boxed{e^{\int 1\,dx} = e^x}$

$\frac{d}{dx}\left[e^x \cdot y\right] = e^x f(x)$

$\frac{d}{dx}\left[e^x \cdot y\right] = e^x$

$e^x \cdot y = e^x + C$

$y = 1 + Ce^{-x}$

$\frac{d}{dx}\left[e^x \cdot y\right] = 0$

$e^x \cdot y = D$

$y = De^{-x}$

$y = \begin{cases} 1 + Ce^{-x} & 0 \le x \le 1 \\ De^{-x} & x > 1 \end{cases}$

<u>$y(0) = 0$</u>: $0 = 1 + Ce^0$

$C = -1$

<u>cont. y</u> $y(1) = y(1)$

$1 - e^{-1} = De^{-1}$

$e - 1 = D$

$y = \begin{cases} 1 - e^{-x} & 0 \le x \le 1 \\ (e-1)e^{-x} & x > 1 \end{cases}$