Warm-up Problems

Solve the IVP $\left(x^2+1\right)\frac{dy}{dx}+2xy=6x,\ y\left(0\right)=5.$ Give the largest interval over which the solution is defined.

Exact Equations

Recall that we defined the differential of $z = f(x,y)$ as $dz = f_x\, dx + f_y\, dy$

Ex. Consider $x^2 - 5xy + y^3 = c$, find the differential of both sides.

$$\underbrace{(2x-5y)}_{f_x}dx + \underbrace{(-5x+3y^2)}_{f_y}dy = 0$$

$M(x,y)dx + N(x,y)dy$ is an <u>exact differential</u> if it is the differential of some function $f(x,y)$. A DE of the form $M(x,y)dx + N(x,y)dy = 0$ is called an <u>exact equation</u>.

→But how can we identify an exact equation?

<u>Thm.</u> $M(x,y)dx + N(x,y)dy$ is an exact differential iff

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

$\langle f_x, f_y \rangle$

$f(x,y)$

Ex. Solve $\overset{M}{2xy}\,dx + (\overset{N}{x^2 - 1})dy = 0.$

$N_x = 2x$ $\therefore$ exact

$M_y = 2x$

$f_x = 2xy$ $\qquad$ $f_y = x^2 - 1$

$f = x^2 y$ $\qquad$ $f = x^2 y - y$

$$x^2 y - y = C$$

Ex. Solve

$$(e^{2y} - y\cos xy)dx + (2xe^{2y} - x\cos xy + 2y)dy = 0.$$

$N_x = 2e^{2y} - [x(-\sin xy \cdot y) + \cos xy \cdot 1]$

$M_y = e^{2y} \cdot 2 - [y(-\sin xy) \cdot x + \cos xy \cdot 1]$

$\therefore$ exact

$f_x = e^{2y} - y\cos xy$

$f = xe^{2y} - \sin xy$

$f_y = 2xe^{2y} - x\cos xy + 2y$

$f = xe^{2y} - \sin xy + y^2$

$$\boxed{xe^{2y} - \sin xy + y^2 = C}$$

$\cos(3x)$

$\frac{1}{3}\sin(3x)$

$\underline{\text{Ex.}}$ Solve $\dfrac{dy}{dx} = \dfrac{xy^2 - \cos x \sin x}{y(1-x^2)}$, $y(0) = 2$

$$y(1-x^2)\,dy = (xy^2 - \cos x \sin x)\,dx$$

$$(\underbrace{\cos x \sin x - xy^2}_{M})\,dx + (\underbrace{y - x^2 y}_{N})\,dy = 0$$

$N_x = -2xy$

$\therefore$ exact

$M_y = -2xy$

$f_x = \cos x \sin x - xy^2$ $\qquad$ $f_y = y - x^2 y$

$f = \frac{1}{2}\sin^2 x - \frac{1}{2}x^2y^2$ $\qquad$ $f = \frac{1}{2}y^2 - \frac{1}{2}x^2y^2$

$$\frac{1}{2}\sin^2 x - \frac{1}{2}x^2y^2 + \frac{1}{2}y^2 = C$$

$$\frac{1}{2}\sin^2 0 - \frac{1}{2}\cdot 0^2 \cdot 2^2 + \frac{1}{2}\cdot 2^2 = C \longrightarrow C = 2$$

$$\frac{1}{2}\sin^2 x - \frac{1}{2}x^2y^2 + \frac{1}{2}y^2 = 2$$

To make $M(x,y)dx + N(x,y)dy$ exact, we multiply by an integrating factor:

If $\dfrac{M_y - N_x}{N}$ is in terms of only x, use $e^{\int \frac{M_y - N_x}{N} dx}$

If $\dfrac{N_x - M_y}{M}$ is in terms of only y, use $e^{\int \frac{N_x - M_y}{M} dy}$

Ex. Solve $xy\,dx + (2x^2 + 3y^2 - 20)dy = 0$.

$N_x = 4x$

$M_y = x$

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y}$$

$$\boxed{e^{\int \frac{3}{y} dy} = e^{3 \ln y} = y^3}$$

$$\left[xy\,dx + (2x^2 + 3y^2 - 20)dy = 0\right] y^3$$

$$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)dy = 0$$

$f_x = xy^4$

$f = \frac{1}{2}x^2y^4$

$f_y = 2x^2y^3 + 3y^5 - 20y^3$

$f = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4$

$$\boxed{\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = C}$$