Some Substitutions

If $\frac{dy}{dx} = f(x)g(y)$, we separate the variables.

If $\frac{dy}{dx} + P(x)y = Q(x)$, we multiply by $e^{\int P(x)dx}$

If $M(x,y)dx + N(x,y)dy = 0$ and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$, we find the potential function.

If $f(tx,ty) = t^{\alpha} f(x,y)$ for some α, we say that f is homogeneous of degree α.

Ex. $f(x,y) = x^3 + x^2y$

$$f(tx,ty) = (tx)^3 + (tx)^2(ty) = t^3x^3 + t^3x^2y = t^3(x^3 + x^2y)$$
$$= t^3 f(x,y)$$

$M(x,y)dx + N(x,y)dy = 0$ is homogeneous if M and N are homogeneous of the same degree.

If we substitute $y = ux$ or $x = vy$ into a homogenous DE, we can reduce it to a separable equation.

<u>Ex.</u> $(x^2+y^2)dx+(x^2-xy)dy=0$

$y=ux \rightarrow u=\frac{y}{x}$

$dy = u\,dx + x\,du$

$(x^2+(ux)^2)dx + (x^2 - x\cdot ux)(u\,dx + x\,du) = 0$

$dx(x^2 + \cancel{u^2x^2} + ux^2 - \cancel{u^2x^2}) + du(x^3 - ux^3) = 0$

$dx(x^2+ux^2) = du(ux^3 - x^3)$

$\dfrac{dx\cdot x^2(\cancel{u+1})}{x^3\,\cancel{u+1}} = \dfrac{du\cdot \cancel{x^3}(u-1)}{\cancel{x^3}\,(u+1)}$

$$\begin{array}{r} 1 - \frac{2}{u+1} \\ u+1\overline{)\,u-1\,} \\ -\;u+1 \\ \hline -2 \end{array}$$

$\int \frac{1}{x}dx = \int \frac{u-1}{u+1}du$

$\int \frac{1}{x}dx = \int 1 - \frac{2}{u+1}du$

$\ln|x| = u - 2\ln|u+1| + C$

$\ln|x| = \frac{y}{x} - 2\ln\left|\frac{y}{x}+1\right| + C$

The DE $\frac{dy}{dx} + P(x)y = f(x)y^n$ is called a Bernoulli equation.

→ Substituting $u = y^{1-n}$ will reduce the DE to a linear equation.

Ex. $x\frac{dy}{dx} + y = x^2 y^2$

$$\left[x\left(-\frac{1}{u^2}\frac{du}{dx}\right) + \frac{1}{u} = x^2\left(\frac{1}{u}\right)^2 \right] \frac{-u^2}{x}$$

$$\frac{du}{dx} - \frac{1}{x}u = -x$$

$$\int \frac{d}{dx}\left[x^{-1}\cdot u\right] = \int -1$$

$$x^{-1}\cdot u = -x + C$$

$$u = -x^2 + Cx$$

$$\frac{1}{y} = -x^2 + Cx$$

$$u = y^{1-2}$$

$$u = \frac{1}{y}$$

$$y = \frac{1}{u} = u^{-1}$$

$$\frac{dy}{dx} = -u^{-2}\frac{du}{dx}$$

$$e^{\int -\frac{1}{x}dx} = e^{-\ln x} = x^{-1}$$

For the equation $\frac{dy}{dx} = f\left(Ax + By + C\right)$, the substitution $u = Ax + By + C$ reduced the DE to separable.

Ex. $\frac{dy}{dx} = (-2x+y)^2 - 7$

$$\frac{du}{dx} + 2 = u^2 - 7$$

$$\frac{du}{dx} = u^2 - 9$$

$$\frac{1}{u^2-9}\,du = dx$$

$$\int \frac{-1/6}{u+3} + \frac{1/6}{u-3}\,du = \int dx$$

$$\frac{1}{6}\left[-\ln|u+3| + \ln|u-3|\right] = x + C$$

$$\frac{1}{6}\left[-\ln|-2x+y+3| + \ln|-2x+y-3|\right] = x + C$$

$$u = -2x + y$$

$$\frac{du}{dx} = -2 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{du}{dx} + 2$$

$$\frac{1}{(u+3)(u-3)} = \frac{A}{u+3} + \frac{B}{u-3}$$

$$1 = A(u-3) + B(u+3)$$

$u = 3$: $1 = B(6) \rightarrow B = 1/6$

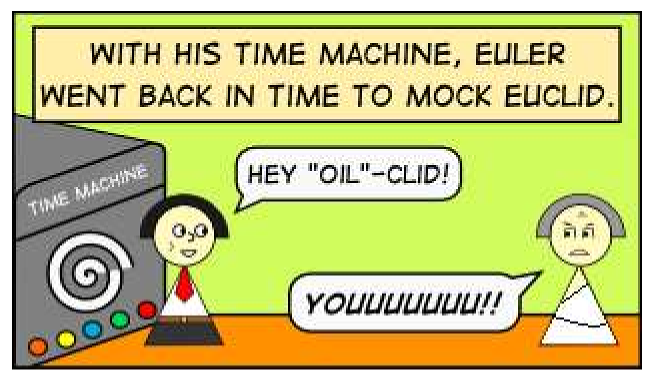
$u = -3$: $1 = A(-6) \rightarrow A = -1/6$

A Numerical Approach

What if we can't solve a 1st order DE algebraically (i.e. none of our other methods work)?

→The slope field gave us an idea of what the solution curve looked like.

→Euler's method will let us approximate values of the solution.



Euler's Method

Starting at the initial value, find the equation of the tangent to the solution at that point.

Follow the tangent line from the initial point for a short interval (h). The point at which you end up is your new starting point, and you begin the process over.

Here's a demonstration.

<u>Ex.</u> Consider the IVP $\dfrac{dy}{dx} = 3xy,\ y(1) = 1$. Use Euler's Method with two steps of equal size to approximate $y(1.4)$.

$$\boxed{y = y_0 + m(x - x_0)}$$

$x = 1 \qquad x = 1.2 \qquad x = 1.4$

$x_0 = 1 \qquad y_0 = 1 \qquad m_0 = 3$

$y = 1 + 3(x - 1)$

$x_1 = 1.2 \qquad y_1 = 1 + 3(1.2 - 1) = 1 + 3(.2) = 1.6 \qquad m_1 = 3(1.2)(1.6) = 5.76$

$y = 1.6 + 5.76(x - 1.2)$

$x_2 = 1.4 \qquad y_2 = 1.6 + 5.76(1.4 - 1.2) = 1.6 + 5.76(.2) = 2.752$

Ex. Redo the previous problem, using four steps of equal size.

$$\frac{dy}{dx} = 3xy$$

from DE

$y_2 = y_1 + y' \cdot \Delta x$

ADD

Δx	x_1	y_1	y'	y_2
	1	1	$3 \cdot 1 \cdot 1 = 3$	$1 + 3(.1) = 1.3$
.1	1.1	1.3	$3(1.1)(1.3) = 4.29$	$1.3 + 4.29(.1) = 1.729$
.1	1.2	1.729	6.224	2.351
.1	1.3	2.351	9.171	3.269
.1	1.4	3.269		