Warm-up Problems

1. Solve the homogeneous DE

$$(y-x)\,dx+(x+y)\,dy=0$$

2. Solve the Bernoulli equation

$$x^2\frac{dy}{dx}+2xy=5y^4$$

Linear Models

<u>Ex.</u> The rate of growth of bacteria in a culture is proportional to the population $P(t)$ of the bacteria after t hours. After 1 hour, the population has increased by 50%. Find the tripling time.

$P(0) = 100$
$P(1) = 150$
$P(?) = 300$

$\frac{dP}{dt} = kP$

$\int \frac{1}{P}\, dP = \int k\, dt$

$\ln|P| = kt + C$

$|P| = e^{kt+C}$

$P = De^{kt}$

$P(0) = De^{0} = 100$

$D = 100$

$P(1) = 100\, e^{k\cdot 1} = 150$

$e^{k} = 1.5$

$k = \ln(1.5)$

$P = 100\, e^{t\ln(1.5)} = 100\, e^{\ln(1.5^t)}$

$P = 100(1.5)^t$

$300 = 100(1.5)^t$

$3 = 1.5^t$

$\ln 3 = \ln(1.5)^t$

$\ln 3 = t \ln 1.5$

$t = \frac{\ln 3}{\ln 1.5}$

Ex. When a cake is removed from an oven, its temperature is 300ºF. Three minutes later, its temperature is 200ºF. Write an equation for the temperature if the room temperature is 70ºF.

$T(0) = 300$
$T(3) = 200$

$\frac{dT}{dt} = k(T - T_m)$

$\frac{dT}{dt} = k(T - 70)$

$\frac{1}{T-70}\, dT = k\, dt$

$\ln|T-70| = kt + C$

$|T - 70| = e^{kt+C}$

$T - 70 = De^{kt}$

$T = 70 + De^{kt}$

$T(0) = 70 + De^0 = 300$

$D = 230$

$T(3) = 70 + 230e^{3k} = 200$

$230e^{3k} = 130$

$e^{3k} = \frac{13}{23}$

$3k = \ln\frac{13}{23}$

$k = \frac{1}{3}\ln\frac{13}{23}$

$T = 70 + 230e^{\frac{t}{3}\ln\frac{13}{23}}$

Ex. A tank initially holds 300 gallons of brine, made up of 50 pounds of salt. A solution with a concentration of 2 pounds of salt per gallon is pumped in at a rate of 3 gallons per minute. While being stirred, fluid is being pumped out at the same rate. Find an equation for the amount of salt in the tank at time *t*.

$$\frac{dA}{dt} = \frac{3\,\cancel{gal}}{min}\cdot\frac{2\text{ lb}}{\cancel{gal}} - \frac{3\text{ gal}}{min}\cdot\frac{A\text{ lb}}{300\text{ gal}}$$

$$\frac{dA}{dt} = 6 - \frac{A}{100}$$

$$\frac{dA}{dt} + \frac{1}{100}A = 6$$

$$\int \frac{d}{dt}\left[e^{\frac{1}{100}t}\cdot A\right] = \int 6e^{\frac{1}{100}t}$$

$$e^{\frac{1}{100}t}\cdot A = 600e^{\frac{1}{100}t} + C$$

$$A = 600 + Ce^{-t/100}$$

$$A(0) = 50$$

$$e^{\int \frac{1}{100}dt} = e^{\frac{1}{100}t}$$

$$A(0) = 600 + Ce^{0} = 50$$

$$C = -550$$

$$A = 600 - 550e^{-t/100}$$

as $t \to \infty$, $A \to \dfrac{600\text{ lb.}}{300\text{ gal.}}$

2 lb/gal.

Kirchhoff's Second Law (circuits):

$$L\frac{di}{dt} + Ri = E(t) \qquad R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

$$\frac{dq}{dt} = i$$

$$\begin{cases} i(t) = \text{current} \\ q(t) = \text{charge} \\ E(t) = \text{voltage} \\ L = \text{inductance} \\ R = \text{resistance} \\ C = \text{capacitance} \end{cases}$$

Ex. A 12-volt battery is connected to a series circuit in which inductance is ½ henry and the resistance is 10 ohms. Determine the current i if the initial current is zero.

$i(0)=0$

$e^{\int 20\,dt} = e^{20t}$

$$L\frac{di}{dt} + Ri = E$$

$$\frac{1}{2}\frac{di}{dt} + 10i = 12$$

$$\frac{di}{dt} + 20i = 24$$

$$\int \frac{d}{dx}\left[e^{20t}\cdot i\right] = \int 24e^{20t}$$

$$e^{20t}\cdot i = \frac{6}{5}e^{20t} + C$$

$$i = \frac{6}{5} + Ce^{-20t}$$

$$i(0) = \frac{6}{5} + Ce^{0} = 0$$

$$C = -\frac{6}{5}$$

$$i = \frac{6}{5} - \frac{6}{5}e^{-20t}$$