Second Order Linear DE's

An IVP would look like

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

$$y(x_0) = y_0,\ y'(x_0) = y_1$$

Thm. Existence of a Unique Solution

Let a_0, a_1, a_2, and $g(x)$ be continuous on an interval containing x_0, and let $a_2(x) \neq 0$ for all x on an interval. Then a unique solution to the IVP exists on the interval.

A Boundary Valued Problem looks like

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

$$y(a) = y_0,\ y(b) = y_1$$

→Could be values of y' at one or both points

→Boundary valued problems do not have to have unique solutions

→The interval of definition would be $[a,b]$

Ex. $y = c_1x^2 + c_2x^4 + 3$ is a solution to $x^2y'' - 5xy' + 8y = 24$. Find a particular solution that satisfies the boundary condition:

a) $y(-1) = 0$, $y(1) = 4$

$y(-1) = c_1 + c_2 + 3 = 0$

$y(1) = c_1 + c_2 + 3 = 4$

$0 = 4$

no solution

Ex. $y = c_1x^2 + c_2x^4 + 3$ is a solution to $x^2y'' - 5xy' + 8y = 24$. Find a particular solution that satisfies the boundary condition:

b) $y(1) = 3$, $y(2) = 15$

$y(1) = c_1 + c_2 + 3 = 3 \longrightarrow c_2 = -c_1$

$y(2) = 4c_1 + 16c_2 + 3 = 15$

$4c_1 - 16c_1 = 12$

$-12c_1 = 12$

$c_1 = -1$

$c_2 = 1$

$y = -x^2 + x^4 + 3$

Ex. $y = c_1x^2 + c_2x^4 + 3$ is a solution to $x^2y'' - 5xy' + 8y = 24$. Find a particular solution that satisfies the boundary condition:

c) $y(0) = 3,\ y(1) = 0$

The DE $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$ is called <u>homogeneous</u> if $g(x) = 0$.

→ For a nonhomogenous equation, we can find the associated homogeneous equation by replacing $g(x)$ with 0.

$$x^2y'' - 5xy' + 8y = 0$$

<u>Thm.</u> Superposition Principle – Homogeneous

Let $y_1, y_2, \ldots, y_k$ be solutions to a homogeneous n^{th} order DE, then

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_k y_k$$

(where $c_1, c_2, \ldots, c_k$ are arbitrary constants) is also a solution.

→ Note that $y = 0$ is always a solution to a homogeneous DE.

<u>Ex.</u> $y_1 = x^2$ and $y_2 = x^2 \ln x$ are solutions to $x^3 y''' - 2xy' + 4y = 0$.

$y_3 =$

$y = c_1 x^2 + c_2 x^2 \ln x + c_3 (\quad)$

A set of functions $f_1, f_2, \ldots, f_n$ is linearly dependent if there exist constants $c_1, c_2, \ldots, c_n$ (not all zero) such that

$$c_1 f_1 + c_2 f_2 + \ldots + c_n f_n = 0$$

→ The set is linearly independent if $c_1 = c_2 = \ldots = c_n = 0$ is the only solution.

→ When there are only 2 functions, they are linearly dependent iff one is a constant multiple of the other.

Ex. Show that $f_1 = \cos^2 x$, $f_2 = \sin^2 x$, $f_3 = \tan^2 x$, and $f_4 = \sec^2 x$ are linearly dependent.

$$\underline{1}\cos^2 x + \underline{1}\sin^2 x + \underline{1}\tan^2 x + \underline{-1}\sec^2 x = 0$$

$$\underbrace{\cos^2 x + \sin^2 x}_{1} \qquad \underbrace{\tan^2 x - \sec^2 x}_{-1}$$

$$\underline{\ \ }f_1 + \underline{\ \ }f_2 + \underline{\ \ }f_3 + \underline{\ \ }f_4 = 0$$

Suppose $f_1, f_2, \ldots, f_n$ possess at least $n - 1$ derivatives. The determinant

$$W = \begin{vmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ f_1' & f_2' & f_3' & \cdots & f_n' \\ f_1'' & f_2'' & f_3'' & \cdots & f_n'' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the <u>Wronskian</u>.

Thm. Let y_1, y_2, …, y_n be solutions of a homogeneous n^{th} order DE, defined on an interval. The set of solutions is linearly independent if $W(y_1, y_2, \ldots, y_n) \neq 0$ for all x on the interval.

Any set $y_1, y_2, \ldots, y_n$ of n linearly independent solutions of a homogeneous n^{th} order DE is called a fundamental set of solutions. This set will always exist.

→To show that solutions form a fundamental set, you must show that they satisfy the equation and that they are independent.

→The general solution to the DE is

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$$

Ex. $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are solutions to
$y'' - 9y = 0$.

Show fund. set

1) Solutions

$y_1 = e^{3x}$
$y_1' = 3e^{3x}$
$y_1'' = 9e^{3x}$

$y'' - 9y = 0$
$9e^{3x} - 9e^{3x} = 0$ ✓

$y_2 = e^{-3x}$
$y_2' = -3e^{-3x}$
$y_2'' = 9e^{-3x}$

$y'' - 9y = 0$
$9e^{-3x} - 9e^{-3x} = 0$ ✓

2) indep.

$y_2 \neq C y_1$
$e^{-3x} \neq C e^{3x}$
$e^{-6x} \neq C$ ✓

$$y = C_1 e^{3x} + C_2 e^{-3x}$$

Ex. $y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$ are solutions to $y''' - 6y'' + 11y' - 6y = 0$.

1) Solutions ✓

2) Indep.

$$W = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^x \begin{vmatrix} 2e^{2x} & 3e^{3x} \\ 4e^{2x} & 9e^{3x} \end{vmatrix} - e^{2x} \begin{vmatrix} e^x & 3e^{3x} \\ e^x & 9e^{3x} \end{vmatrix} + e^{3x} \begin{vmatrix} e^x & 2e^{2x} \\ e^x & 4e^{2x} \end{vmatrix}$$

$$= e^x(18e^{5x} - 12e^{5x}) - e^{2x}(9e^{4x} - 3e^{4x}) + e^{3x}(4e^{3x} - 2e^{3x})$$

$$= 6e^{6x} - 6e^{6x} + 2e^{6x} = 2e^{6x} \neq 0$$ ✓

$$y = C_1 e^x + C_2 e^{2x} + C_3 e^{3x}$$

Any function y_p that is free of parameters and satisfies a non–homogeneous DE is called a particular solution.

Ex. $y = 3$ is a particular solution of
$y'' + 9y = 27$.

Homogeneous

(*) $$a_n(x)y^{(n)} + \ldots + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

Non–Homogeneous

(**) $$a_n(x)y^{(n)} + \ldots + a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

If $y_1, y_2, \ldots, y_k$ are solutions of (*) and y_p is a solution to (**), then

$$y = \underbrace{c_1y_1 + c_2y_2 + \ldots + c_ky_k}_{} + y_p$$

is a solution to (**).

Thm. Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions to (*), and let y_p be a solution to (**), then

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n + y_p$$

is the general solution to (**).

$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ is called the complementary function of (**), written y_c.

→ So the general solution can be written $y_c + y_p$.

Ex. Find the general solution to

$$y''' - 6y'' + 11y' - 6y = 3.$$

1) $y''' - 6y'' + 11y' - 6y = 0$

$y_c = C_1 e^{x} + C_2 e^{2x} + C_3 e^{3x}$

2) $y''' - 6y'' + 11y' - 6y = 3$

$y_p = -\frac{1}{2}$

$$y = C_1 e^{x} + C_2 e^{2x} + C_3 e^{3x} - \frac{1}{2}$$

Thm. Superposition Principle – Nonhomogeneous

$$\left.\begin{aligned} a_n y^{(n)} + \ldots + a_2 y'' + a_1 y' + a_0 y &= g_1(x) \leftarrow y_{p_1} \\ a_n y^{(n)} + \ldots + a_2 y'' + a_1 y' + a_0 y &= g_2(x) \leftarrow y_{p_2} \\ &\vdots \\ a_n y^{(n)} + \ldots + a_2 y'' + a_1 y' + a_0 y &= g_k(x) \leftarrow y_{p_k} \end{aligned}\right\} \text{particular solutions}$$

then $y_p = y_{p_1} + y_{p_2} + \ldots + y_{p_k}$ is a particular solution of

$$a_n y^{(n)} + \ldots + a_2 y'' + a_1 y' + a_0 y = g_1 + g_2 + \ldots + g_k$$

Ex.

$y_{p_1} = -4x^2$ is a solution to $y'' - 3y' + 4y = -16x^2 + 24x - 8$

$y_{p_2} = e^{2x}$ is a solution to $y'' - 3y' + 4y = 2e^{2x}$

$y_{p_3} = xe^x$ is a solution to $y'' - 3y' + 4y = 2xe^x - e^x$

$$y'' - 3y' + 4y = (-16x^2 + 24x - 8) + (2e^{2x}) + (2xe^x - e^x)$$

$$y_p = -4x^2 + e^{2x} + xe^x$$

$$y'' - 3y' + 4y = 2(-16x^2 + 24x - 8) + \tfrac{1}{3}(2e^{2x}) - (2xe^x - e^x)$$

$$y_p = 2(-4x^2) + \tfrac{1}{3}(e^{2x}) - (xe^x)$$