Warm-up Problems

Consider the functions $y_1 = \sin 2x$ and $y_2 = \cos 2x$, and the DE $y'' + 4y = 0$.

1) Show that y_1 and y_2 form a fundamental set of solutions.

2) Write the general solution.

3) Find a member of the family that satisfies the boundary conditions $y(0) = 2$ and $y'\left(\frac{\pi}{2}\right) = 0$, if possible.

Reduction of Order

Suppose you know that y_1 is a solution to $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, and you want to find the other solution y_2 to form the fundamental set of solutions.

Because they are linearly independent, $y_2 \neq cy_1$ for constant c.

→$y_2 = u(x)y_1$ for some function $u(x)$.

→We can make this substitution to find y_2.

Ex. If $y_1 = e^x$ is a solution of $y'' - y = 0$, use reduction of order to find a second solution.

$y_2 = u \cdot e^x$

$y_2' = ue^x + e^x u'$

$y_2'' = ue^x + e^x u' + e^x u'' + u' e^x$

$= e^x(u + 2u' + u'')$

$v = u'$

$v' = u''$

$e^{\int 2dx} = e^{2x}$

$e^x(u + 2u' + u'') - ue^x = 0$

$e^x(u'' + 2u') = 0$

$u'' + 2u' = 0$

$v' + 2v = 0$

$\frac{d}{dx}\left[v \cdot e^{2x}\right] = 0$

$v \cdot e^{2x} = C$

$v = Ce^{-2x} \leftarrow u'$

$u = De^{-2x} + F$

$y_2 = (De^{-2x})e^x$

$y_2 = e^{-x}$

For the DE $y'' + P(x)y' + Q(x)y = 0$,

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1)^2} dx$$

On the homework, do problems 1–3 without the formula.

Ex. If $y_1 = x^2$ is a solution of
$x^2y'' - 3xy' + 4y = 0$, use ~~the~~ reduction of order ~~formula~~ to find a second solution.

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

$$y_2 = x^2\int \frac{e^{-\int -\frac{3}{x}dx}}{(x^2)^2}dx = x^2\int \frac{e^{3\ln x}}{x^4}dx = x^2\int \frac{x^3}{x^4}dx$$

$$= x^2\int \frac{1}{x}dx = \boxed{x^2\ln x}$$

Ex. If $y_1 = e^x$ is a solution of $y'' - 4y' + 3y = 0$, find the general solution to $y'' - 4y' + 3y = x$.

Homog. $y'' - 4y' + 3y = 0$

$$y_2 = e^x \int \frac{e^{-\int -4\,dx}}{(e^x)^2}\,dx = e^x \int \frac{e^{4x}}{e^{2x}}\,dx = e^x \int e^{2x}\,dx = e^x\left(\tfrac{1}{2}e^{2x}\right)$$

$$\Rightarrow y_2 = e^{3x}$$

$$y_c = C_1 e^x + C_2 e^{3x}$$

Non-homog. $y'' - 4y' + 3y = x$

$y_p = Ax + B$

$y_p' = A$

$y_p'' = 0$

$$0 - 4(A) + 3(Ax + B) = x$$

$$3Ax + (-4A + 3B) = x$$

$3A = 1$

$A = \frac{1}{3}$

$-4A + 3B = 0$

$-\frac{4}{3} + 3B = 0$

$3B = \frac{4}{3} \rightarrow B = \frac{4}{9}$

$$y = C_1 e^x + C_2 e^{3x} + \frac{1}{3}x + \frac{4}{9}$$