Warm-up Problem

The function $y_1 = e^{x/2}$ is a solution to the differential equation $4y'' - 4y' + y = 0$. Use reduction of order to find a second solution.

Homog. Equations with Constant Coefficients

~~Consider~~ $ay'' + by' + cy = 0$. A solution will have the form $y = e^{mx}$ $\longrightarrow y' = e^{mx} \cdot m \rightarrow y'' = e^{mx} \cdot m^2$

$$a(m^2 e^{mx}) + b(me^{mx}) + c(e^{mx}) = 0$$

$$e^{mx}(am^2 + bm + c) = 0$$

$$am^2 + bm + c = 0$$

This is called the <u>auxiliary equation</u>.

3 Cases

If distinct roots m_1 and m_2 → $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

If repeating root m → $y = c_1 e^{mx} + c_2 x e^{mx}$

If complex conjugate roots $\alpha \pm \beta i$ →

$$y = e^{\alpha x}\left(c_1 \cos \beta x + c_2 \sin \beta x\right)$$

Ex. Solve $2y'' - 5y' - 3y = 0$

$$2m^2 - 5m - 3 = 0$$

$$(2m+1)(m-3) = 0$$

$$m = -\frac{1}{2}, \; 3$$

$$y = C_1 e^{-\frac{1}{2}x} + C_2 e^{3x}$$

Ex. Solve $y'' - 10y' + 25y = 0$

$$m^2 - 10m + 25 = 0$$

$$(m-5)^2 = 0$$

$$m = 5$$

$$y = C_1 e^{5x} + C_2 x e^{5x}$$

<u>Ex.</u> Solve $y'' + 4y' + 7y = 0$

$$m^2 + 4m + 7 = 0$$

$$m = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 7}}{2 \cdot 1} = \frac{-4 \pm \sqrt{16 - 28}}{2} = \frac{-4 \pm \sqrt{-12}}{2} = \frac{-4 \pm 2i\sqrt{3}}{2}$$

$$= -2 \pm i\sqrt{3}$$

$$y = e^{-2x}\left(C_1 \cos(\sqrt{3}\,x) + C_2 \sin(\sqrt{3}\,x)\right)$$

Ex. Solve the IVP $4y'' + 4y' + 17y = 0$,
$y(0) = -1,\ y'(0) = 2$

$$4m^2 + 4m + 17 = 0$$

$$m = \frac{-4 \pm \sqrt{4^2 - 4\cdot4\cdot17}}{2\cdot4} = \frac{-4 \pm \sqrt{16-272}}{8} = \frac{-4 \pm \sqrt{-256}}{8} = \frac{-4 \pm 16i}{8} = -\frac{1}{2} \pm 2i$$

$$y = e^{-\frac{1}{2}x}(C_1\cos 2x + C_2 \sin 2x)$$

$$y(0) = e^0(C_1\cdot 1 + C_2 \cdot 0) = -1 \rightarrow C_1 = -1$$

$$y' = e^{-\frac{1}{2}x}(-2C_1 \sin 2x + 2C_2\cos 2x) + \left(-\tfrac{1}{2}e^{-\frac{1}{2}x}\right)(C_1\cos 2x + C_2 \sin 2x)$$

$$y'(0) = e^0(0 + 2C_2\cdot 1) - \tfrac{1}{2}e^0(C_1 + 0) = 2$$

$$2C_2 + \tfrac{1}{2} = 2 \rightarrow 2C_2 = \tfrac{3}{2} \rightarrow C_2 = \tfrac{3}{4}$$

$$\boxed{y = e^{-\frac{1}{2}x}\left(-\cos 2x + \tfrac{3}{4}\sin 2x\right)}$$

Useful to remember:

$y'' + k^2 y = 0$ has the solution

$$y = c_1 \cos kx + c_2 \sin kx$$

This works for higher order equations, but it's harder to break down the cases:

If all roots are distinct m_1, m_2, …, m_n →

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$$

If a root m repeats k times →

$$y = c_1 e^{mx} + c_2 x e^{mx} + c_3 x^2 e^{mx} + \ldots + c_k x^{k-1} e^{mx}$$

Remember, if there is a complex root, then its conjugate is also a root: $\alpha \pm \beta i$

→ A cubic polynomial must have at least one real root.

Ex. Solve $y''' + 3y'' - 4y = 0$

$$m^3 + 3m^2 - 4 = 0$$

$$\boxed{y = C_1 e^{x} + C_2 e^{-2x} + C_3 x e^{-2x}}$$

$$(m-1)(m^2+4m+4) = 0$$

$$(m-1)(m+2)^2 = 0$$

$$m = 1, -2$$

$$\begin{array}{r|rrrr}
1 & 1 & 3 & 0 & -4 \\
  &   & 1 & 4 & 4 \\
\hline
  & 1 & 4 & 4 & \boxed{0}
\end{array}$$

$$\begin{array}{r}
m^2 + 4m + 4 \\
m-1 \overline{)\, m^3 + 3m^2 \qquad - 4} \\
-(m^3 - m^2) \qquad\qquad \\
\hline
4m^2 \qquad\qquad \\
-(4m^2 - 4m) \qquad \\
\hline
4m - 4 \\
-(4m - 4) \\
\hline
0
\end{array}$$

Ex. Solve $y'''' + 2y'' + y = 0$

$\boxed{y^{(4)}}$

$m^4 + 2m^2 + 1 = 0 \longrightarrow$ $p = m^2$

$p^2 + 2p + 1$

$(p+1)^2$

$(m^2+1)^2$

$(m^2+1)^2 = 0$

$m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$

$y = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x$

$(C_1 \cos x + C_2 \sin x) + x(C_3 \cos x + C_4 \sin x)$

Ex. Solve $y''' - 3y'' = 0$

$$m^3 - 3m^2 = 0$$

$$m^2(m-3) = 0$$

$$m = 0,\ 3$$

$$y = C_1 e^{0x} + C_2 x e^{0x} + C_3 e^{3x}$$

$$y = C_1 + C_2 x + C_3 e^{3x}$$