Undetermined Coefficients — Superposition

Consider the problem of finding y_p for the non–homogeneous DE

$$a_n y^{(n)} + \ldots + a_2 y'' + a_1 y' + a_0 y = g(x)$$

where a_0, a_1, …, a_n are constant, and where $g(x)$ is either polynomial, exponential, sine, cosine, or the sum or product of these.

$g(x) = 7$ $\qquad$ $g(x) = x^5 - 3x$ $\qquad$ $g(x) = \cos 4x$

$g(x) = e^{7x}$ $\qquad$ $g(x) = \cos 2x - 3xe^x$

$g(x) = xe^{2x} \cos 4x$

Derivatives of functions in this group will remain in the group.

→ This means we can assume that y_p has the same form as $g(x)$.

→ No logarithms, no negative or fractional powers, and no other trig functions

Ex. Find y_p for $y'' - y' + y = 2\sin 3x$

$$y_p = A\sin 3x + B\cos 3x$$

$$y_p' = 3A\cos 3x - 3B\sin 3x$$

$$y_p'' = -9A\sin 3x - 9B\cos 3x$$

$$(-9A\sin 3x - 9B\cos 3x) - (3A\cos 3x - 3B\sin 3x) + (A\sin 3x + B\cos 3x) = 2\sin 3x$$

$$\sin 3x(-9A + 3B + A) + \cos 3x(-9B - 3A + B) = 2\sin 3x$$

$$-8A + 3B = 2 \qquad -3A - 8B = 0$$

$$B = -\frac{3}{8}A$$

$$-8A - \frac{9}{8}A = 2$$

$$-\frac{73}{8}A = 2$$

$$A = -\frac{16}{73}$$

$$B = -\frac{3}{8} \cdot \frac{-16}{73} = \frac{6}{73}$$

$$y_p = \frac{-16}{73}\sin 3x + \frac{6}{73}\cos 3x$$

$\underline{\text{Ex.}}$ Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6$

$\underline{y'' + 4y' - 2y = 0}$

$m^2 + 4m - 2 = 0 \longrightarrow m = \frac{-4 \pm \sqrt{16 - 4(-2)}}{2} = \frac{-4 \pm \sqrt{24}}{2} = \frac{-4 \pm 2\sqrt{6}}{2} = -2 \pm \sqrt{6}$

$y_c = C_1 e^{(-2+\sqrt{6})x} + C_2 e^{(-2-\sqrt{6})x}$

$\underline{y'' + 4y' - 2y = 2x^2 - 3x + 6}$

$y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

$2A + 4(2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 - 3x + 6$

$-2Ax^2 + x(8A - 2B) + (2A + 4B - 2C) = 2x^2 - 3x + 6$

$-2A = 2$

$A = -1 \longrightarrow$

$8A - 2B = -3$

$-8 - 2B = -3$

$-2B = 5$

$B = -\frac{5}{2}$

$2A + 4B - 2C = 6$

$2(-1) + 4(-\frac{5}{2}) - 2C = 6$

$-2 - 10 - 2C = 6$

$-2C = 18$

$C = -9$

$$\boxed{y = C_1 e^{(-2+\sqrt{6})x} + C_2 e^{(-2-\sqrt{6})x} + \left(-x^2 - \frac{5}{2}x - 9\right)}$$

Ex. Solve $y'' - 5y' + 4y = 8e^x$

$y'' - 5y' + 4y = 0$

$m^2 - 5m + 4 = 0$

$(m-4)(m-1) = 0$

$m = 1, 4$

$y_c = C_1 e^x + C_2 e^{4x}$

$y'' - 5y' + 4y = 8e^x$

$$\left.\begin{aligned} y_p &= Ae^x \\ y_p' &= Ae^x \\ y_p'' &= Ae^x \end{aligned}\right\}$$

$Ae^x - 5Ae^x + 4Ae^x = 8e^x$

$Ae^x(1 - 5 + 4) = 8e^x$

$Ae^x \cdot 0 = 8e^x$

Case I: No part of the assumed y_p is part of y_c

→Look at p. 143 for several typical forms we can use

Ex. Find the form of y_p

a) $g(x) = 5x^3e^{-x} - 7e^x$

$y_p = (Ax^3 + Bx^2 + Cx + D)e^{-x} + Fe^x$

b) $g(x) = x\cos x$

$y_p = (Ax + B)\cos x + (Cx + D)\sin x$

c) $g(x) = 3x^2 - 5\sin 2x + 7xe^{6x}$

$y_p = Ax^2 + Bx + C + D\sin 2x + F\cos 2x + (Gx + H)e^{6x}$

Case II: A part of the assumed y_p is part of y_c

Ex. What is the form of y_p for

$$\underbrace{y'' - 6y' + 9y}\ = \underbrace{6x^2 + 2} - 12e^{3x}$$

$m^2 - 6m + 9 = 0$

$(m-3)^2 = 0$

$m = 3$

$y_c = C_1 e^{3x} + C_2 x e^{3x}$

$y_p = Ax^2 + Bx + C + Dx^2 e^{3x}$

$y'' - 6y' + 9y = 6x^2 + 2 \longrightarrow y_p = Ax^2 + Bx + C$

$y'' - 6y' + 9y = -12e^{3x} \longrightarrow y_p = Dx^2 e^{3x}$

→ Multiply by enough x's so that y_p is no longer part of y_c

<u>Ex.</u> Solve $y'' + y = 4x + 10\sin x$, $y(\pi) = 0$, $y'(\pi) = 2$.

$y'' + y = 0 \longrightarrow y_c = C_1 \sin x + C_2 \cos x$

$y_p = Ax + B + Cx\sin x + Dx\cos x$

$y_p' = A + C\sin x + Cx\cos x + D\cos x - Dx\sin x$

$y_p'' = C\cos x + C\cos x - Cx\sin x - D\sin x - D\sin x - Dx\cos x$

$y'' + y = 4x + 10\sin x$

$(2C\cos x - Cx\sin x - 2D\sin x - Dx\cos x) + (Ax + B + Cx\sin x + Dx\cos x) = 4x + 10\sin x$

$\cos x(2C) + \sin x(-2D) + \underbrace{x\cos x(-D+D)} + \underbrace{x\sin x(-C+C)} + Ax + B = 4x + 10\sin x$

$2C = 0$ $\quad -2D = 10$ $\quad A = 4$ $\quad B = 0$

$C = 0$ $\quad D = -5$

$y_p = 4x - 5x\cos x$

$y = C_1 \sin x + C_2 \cos x + 4x - 5x\cos x$

$$y = C_1 \sin x + C_2 \cos x + 4x - 5x\cos x$$

$$y' = C_1 \cos x - C_2 \sin x + 4 - 5\cos x + 5x \sin x$$

$$y(\pi) = C_1 \cdot 0 + C_2(-1) + 4\pi - 5\pi(-1) = 0$$

$$-C_2 + 9\pi = 0 \rightarrow C_2 = 9\pi$$

$$y'(\pi) = C_1(-1) - C_2(0) + 4 - 5(-1) + 5\pi(0) = 2$$

$$-C_1 + 9 = 2 \rightarrow C_1 = 7$$

$$y = 7\sin x + 9\pi \cos x + 4x - 5x\cos x$$

Ex. Find the form of y_p for $y^{(4)} + y''' = 1 - x^2e^{-x}$

$y^{(4)} + y''' = 0$

$m^4 + m^3 = 0$

$m^3(m+1) = 0$

$m = 0, -1$

$y_c = C_1 + C_2x + C_3x^2 + C_4e^{-x}$

$y_p = Ax^3 + (Bx^2 + Cx + D)xe^{-x}$

$Bx^3e^{-x} + Cx^2e^{-x} + De^{-x}x$