Warm–up Problems

1) Find the general solution of $y'' - 5y' + 6y = 8 + 2\sin x$.

2) How would your work change if the equation was $y'' + y = 8 + 2\sin x$?

Variation of Parameters — Another Method for Finding y_p

1) Find $y_c = c_1y_1 + c_2y_2$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

2) Find the Wronskian, W, of y_1 and y_2.

3) Put the DE in the form $y'' + Py' + Qy = f(x)$

4) Find $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$ and $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$

5) Find $u_1 = \int \frac{W_1}{W}dx$ and $u_2 = \int \frac{W_2}{W}dx$

6) $y_p = u_1y_1 + u_2y_2$

<u>Ex.</u> Solve $y'' - 4y' + 4y = (x+1)e^{2x}$

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2$

$y_1 = e^{2x}$

$y_2 = xe^{2x}$

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}+2xe^{2x} \end{vmatrix} = e^{4x} + 2xe^{4x} - 2xe^{4x} = e^{4x}$$

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & e^{2x}+2xe^{2x} \end{vmatrix} = -(x^2+x)e^{4x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}$$

$$u_1 = \int \frac{W_1}{W} = \int \frac{-(x^2+x)e^{4x}}{e^{4x}} = -\frac{1}{3}x^3 - \frac{1}{2}x^2$$

$$u_2 = \int \frac{W_2}{W} = \int \frac{(x+1)e^{4x}}{e^{4x}} = \frac{1}{2}x^2 + x$$

$$y = \underbrace{C_1 e^{2x} + C_2 x e^{2x}}_{y_c} + \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x}$$

Ex. Solve $4y'' + 36y = \csc 3x$ $\longrightarrow$ $y'' + 9y = \frac{1}{4}\csc 3x$

$4m^2 + 36 = 0$

$m^2 = -9$

$m = \pm 3i$

$y_1 = \cos 3x$

$y_2 = \sin 3x$

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3\cos^2 3x + 3\sin^2 3x = 3$$

$$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4}\csc 3x & 3\cos 3x \end{vmatrix} = -\frac{1}{4}$$

$$W_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \frac{1}{4}\csc 3x \end{vmatrix} = \frac{1}{4}\cot 3x$$

$$u_1 = \int \frac{W_1}{W} = \int \frac{-1/4}{3} = -\frac{1}{12}x$$

$$u_2 = \int \frac{W_2}{W} = \int \frac{\frac{1}{4}\cot 3x}{3} = \frac{1}{12}\cdot\frac{1}{3}\ln|\sin 3x|$$

$$y = C_1\cos 3x + C_2\sin 3x - \frac{1}{12}x\cos 3x + \frac{1}{36}\ln|\sin 3x|\sin 3x$$

For higher order:

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix} \qquad W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{vmatrix}$$

Ex. Solve $y''' + 4y' = \sec 2x$

$m^3 + 4m = 0$

$m(m^2+4) = 0$

$m = 0, 2i, -2i$

$y_1 = 1$

$y_2 = \cos 2x$

$y_3 = \sin 2x$

$$W = \begin{vmatrix} 1 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ 0 & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$= 1 \begin{vmatrix} -2\sin 2x & 2\cos 2x \\ -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$= 8\sin^2 2x + 8\cos^2 2x = 8$$

$$W_1 = \begin{vmatrix} 0 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ \sec 2x & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin 2x \\ 0 & 0 & 2\cos 2x \\ 0 & \sec 2x & -4\sin 2x \end{vmatrix}$$

$$W_3 \begin{vmatrix} 1 & \cos 2x & 0 \\ 0 & -2\sin 2x & 0 \\ 0 & -4\cos 2x & \sec 2x \end{vmatrix}$$

$$u_1 = \int \frac{W_1}{W}$$

$$u_2 = \int \frac{W_2}{W}$$

$$u_3 = \int \frac{W_3}{W}$$