Cauchy–Euler Equation

$$a_n x^n y^{(n)} + \ldots + a_2 x^2 y'' + a_1 x y' + a_0 y = g(x)$$

where a_0, a_1, a_2, …, a_n are constant, is called a <u>Cauchy–Euler</u> equation.

→Note that the lead coefficient equals 0 at $x = 0$, so we will restrict the interval to $(0,\infty)$ to ensure a unique solution.

Start ~~with~~ $ax^2y'' + bxy' + cy = 0$, we are going to assume that a solution is $y = x^m$

$$y' = mx^{m-1}$$

$$y'' = m(m-1)x^{m-2}$$

$$ax^2 \cdot m(m-1)x^{m-2} + bx \cdot mx^{m-1} + cx^m = 0$$

$$am(m-1)x^m + bmx^m + cx^m = 0$$

$$x^m\left[am(m-1) + bm + c\right] = 0$$

$$am(m-1) + bm + c = 0$$

This is our auxiliary equation.

3 Cases

If distinct roots m_1 and m_2 → $y = c_1 x^{m_1} + c_2 x^{m_2}$

If repeating root m → $y = c_1 x^m + c_2 x^m \ln x$

If complex conjugate roots $\alpha \pm \beta i$ →

$$y = x^{\alpha}\left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right]$$

Ex. Solve $x^2y'' - 2xy' - 4y = 0$

$m(m-1) - 2m - 4 = 0$

$m^2 - 3m - 4 = 0$

$(m-4)(m+1) = 0$

$m = 4, -1$

$y = C_1 x^4 + C_2 x^{-1}$

Ex. Solve $4x^2y'' + 17y = 0$, $y(1) = -1$,

$y'(1) = -1/2$

$$4m(m-1) + 17 = 0$$
$$4m^2 - 4m + 17 = 0$$

$$\to m = \frac{4 \pm \sqrt{16 - 4\cdot 4\cdot 17}}{2\cdot 4} = \frac{4 \pm \sqrt{16-272}}{8} = \frac{4 \pm \sqrt{-256}}{8} = \frac{4 \pm 16i}{8}$$
$$= \frac{1}{2} \pm 2i$$

$$y = x^{1/2}\left(c_1 \cos(2\ln x) + C_2 \sin(2\ln x)\right)$$

$$y(1) = 1(C_1 \cdot 1 + C_2 \cdot 0) = -1 \to C_1 = -1$$

$$y' = x^{1/2}\left[-C_1 \sin(2\ln x)\cdot \frac{2}{x} + C_2 \cos(2\ln x)\cdot\frac{2}{x}\right] + \frac{1}{2}x^{-1/2}\left[c_1 \cos(2\ln x) + C_2 \sin(2\ln x)\right]$$

$$y'(1) = 1(-C_1 \cdot 0 + C_2 \cdot 1 \cdot 2) + \frac{1}{2}((-1)\cdot 1 + C_2 \cdot 0) = \frac{-1}{2}$$

$$2C_2 - \frac{1}{2} = -\frac{1}{2} \to 2C_2 = 0$$
$$C_2 = 0$$

$$\boxed{y = -x^{1/2}\cos(2\ln x)}$$

<u>Ex.</u> Solve $x^3y''' + 5x^2y'' + 7xy' + 8y = 0$

$m(m-1)(m-2) + 5m(m-1) + 7m + 8 = 0$

$m^3 - 3m^2 + 2m + 5m^2 - 5m + 7m + 8 = 0$

$\underbrace{m^3 + 2m^2} + \underbrace{4m + 8} = 0$

$m^2(m+2) + 4(m+2) = 0$

$(m+2)(m^2+4) = 0$

$m = -2, \pm 2i$

$y = C_1 x^{-2} + C_2 \cos(2\ln x) + C_3 \sin(2\ln x)$

Ex. Solve $x^2y'' - 3xy' + 3y = 2x^4e^x$ $\rightarrow$ $y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x$

$x^2y'' - 3xy' + 3y = 0$

$m(m-1) - 3m + 3 = 0$

$m^2 - 4m + 3 = 0$

$(m-3)(m-1) = 0$

$m = 1, 3$

$y_1 = x$

$y_2 = x^3$

$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 3x^3 - x^3 = 2x^3$

$W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix} = -2x^5e^x$ $\rightarrow$ $u_1 = \int \frac{-2x^5e^x}{2x^3} = \int -x^2e^x = -x^2e^x + 2xe^x - 2e^x$

$W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix} = 2x^3e^x$ $\rightarrow$ $u_2 = \int \frac{2x^3e^x}{2x^3} = \int e^x = e^x$

$y = C_1x + C_2x^3 + (-x^2e^x + 2xe^x - 2e^x)x + e^x x^3$

$\int x^2e^x$

$\boxed{\begin{matrix} u = x^2 & dv = e^x dx \\ du = 2x\,dx & v = e^x \end{matrix}}$

$= x^2e^x - \int 2xe^x dx$

$\boxed{\begin{matrix} u = 2x & dv = e^x dx \\ du = 2dx & v = e^x \end{matrix}}$

$= x^2e^x - \left[2xe^x - \int 2e^x dx\right]$

It’s possible to reduce a Cauchy–Euler equation into a linear equation with constant coefficients using the substitution

$$x = e^t$$

Ex. Solve $x^2y'' - xy' + y = \ln x$