Higher Order Linear Models

Kirchhoff's Second Law (circuits):

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

$$q' \longrightarrow \quad i(t) = \text{current}$$

$$q(t) = \text{charge}$$

$$E(t) = \text{voltage}$$

$$L = \text{inductance}$$

$$R = \text{resistance}$$

$$C = \text{capacitance}$$

Ex. Find the charge $q(t)$ on the capacitor when $L = 0.25$, $R = 10$, $C = 0.01$, $E(t) = 0$, $q(0) = q_0$, and $i(0) = 0$.

$$.25q'' + 10q' + \frac{1}{.01}q = 0$$

$$q'' + 40q' + 400q = 0$$

$$m^2 + 40m + 400 = 0$$

$$(m+20)^2 = 0$$

$$m = -20$$

$$q = C_1 e^{-20t} + C_2 t e^{-20t}$$

$$q(0) = C_1 + 0 = q_0$$

$$q' = -20C_1 e^{-20t} + C_2 e^{-20t} - 20C_2 t e^{-20t}$$

$$q'(0) = -20q_0 + C_2 = 0$$

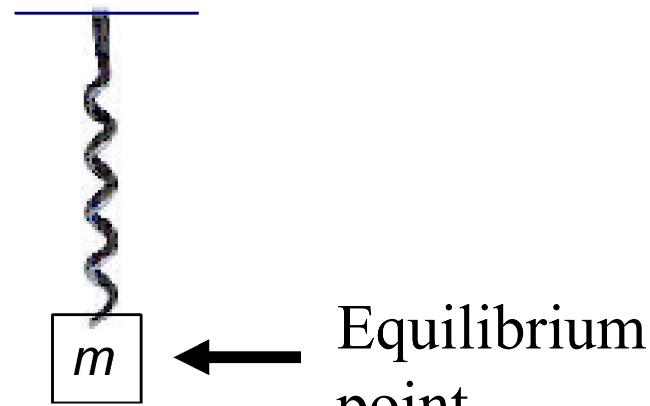
$$C_2 = 20q_0$$

$$q = q_0 e^{-20t} + 20q_0 t e^{-20t}$$

Spring supporting a mass (Free Undamped):

The motion of the mass can be described by

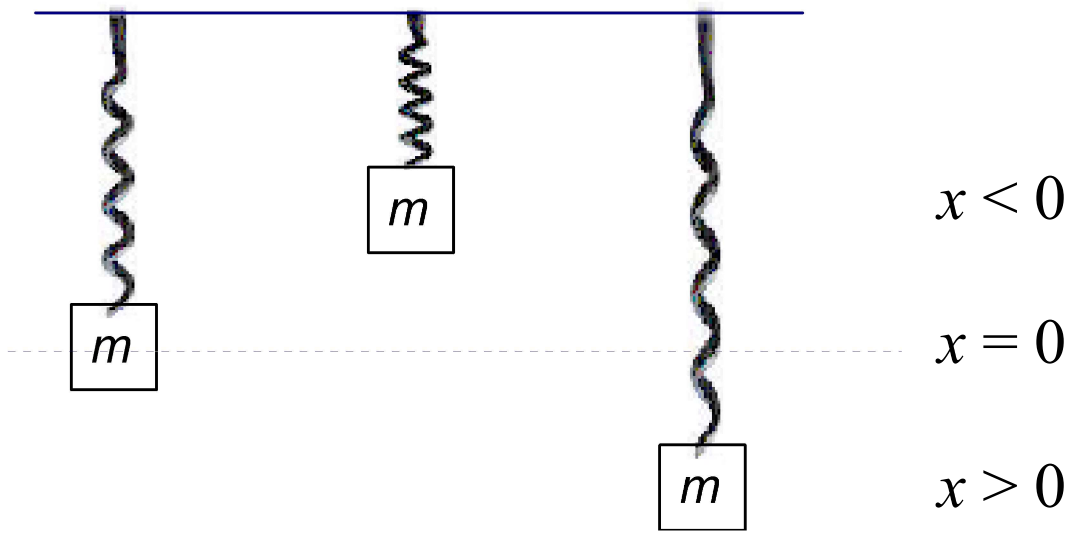
$$m\frac{d^2x}{dt^2} = -kx$$



m = mass

x = displacement from equilibrium (in feet)

k = spring constant



$x' > 0$ means pulled down

$x' < 0$ means pushed up

Starting from rest means $x'(0) = 0$

Mass = pounds/32

Pounds = k(stretch caused by mass)

Ex. A mass weighing 2 lbs. stretches a spring ~~6 in.~~ $\frac{1}{2}$ ft. At $t = 0$, the mass is released from a point ~~8 in.~~ $\frac{2}{3}$ ft. below equilibrium with an upward velocity of $\frac{4}{3}\frac{\text{ft.}}{\text{sec.}}$. Determine the equation of motion.

$$m = \frac{2}{32} = \frac{1}{16}$$

$$2 = k\left(\frac{1}{2}\right) \rightarrow k = 4$$

$$x(0) = \frac{2}{3}$$

$$x'(0) = -\frac{4}{3}$$

$$mx'' = -kx$$

$$\frac{1}{16}x'' = -4x$$

$$x'' + 64x = 0$$

$$m^2 + 64 = 0$$

$$m = \pm 8i$$

$$x = C_1 \cos 8t + C_2 \sin 8t$$

$$x(0) = C_1 + 0 = \frac{2}{3}$$

$$x' = -8C_1 \sin 8t + 8C_2 \cos 8t$$

$$x'(0) = 0 + 8C_2 = -\frac{4}{3}$$

$$C_2 = -\frac{1}{6}$$

$$x = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t$$

Spring supporting a mass (Free Damped):

Motion is being affected by the surrounding environment (air resistance, fluid, etc.)

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}$$

for some constant β.

Ex. A mass weighing 8 lbs. stretches a spring 2 ft. Assuming a damping force numerically equal to 2 times the instantaneous velocity, determine the equation of motion if the mass is released from equilibrium with an upward velocity of 3 ft/s.

$m = \frac{8}{32} = \frac{1}{4}$

$8 = k(2) \rightarrow k = 4$

$\beta = 2$

$x(0) = 0$

$x'(0) = -3$

$\frac{1}{4}x'' = -4x - 2x'$

$x'' + 8x' + 16x = 0$

$m^2 + 8m + 16 = 0$

$(m+4)^2 = 0$

$m = -4$

$x = C_1 e^{-4t} + C_2 t e^{-4t}$

$x(0) = C_1 + 0 = 0$

$C_1 = 0$

$x' = -4C_1 e^{-4t} + C_2 e^{-4t} - 4C_2 t e^{-4t}$

$x'(0) = 0 + C_2 + 0 = -3$

$C_2 = -3$

$x = -3te^{-4t}$

Spring supporting a mass (Driven Motion):

Motion is also being affected by a force $f(t)$ on the support of the spring.

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} + f(t)$$

Ex. A mass of 1 slug is attached to a spring whose constant is 5 lb/ft. Initially, the mass is released 1 ft below equilibrium with a downward velocity of 5 ft/s, and motion is damped by a force numerically equal to 2 times the instantaneous velocity. If motion is driven by an external force $f(t) = 12\cos 2t + 3\sin 2t$, find the equation of motion.

m k $x'(0)$ $x(0)$ β

$$1 \cdot x'' = -5x - 2x' + 12\cos 2t + 3\sin 2t \longrightarrow x'' + 2x' + 5x = 12\cos 2t + 3\sin 2t$$

$$x'' + 2x' + 5x = 0$$
$$m^2 + 2m + 5 = 0 \longrightarrow m = \frac{-2 \pm \sqrt{4 - 4 \cdot 5}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$x_c = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$$

$$x_p = A\cos 2t + B\sin 2t$$
$$x_p' = -2A\sin 2t + 2B\cos 2t$$
$$x_p'' = -4A\cos 2t - 4B\sin 2t$$

$$(-4A\cos 2t - 4B\sin 2t) + 2(-2A\sin 2t + 2B\cos 2t) + 5(A\cos 2t + B\sin 2t) = 12\cos 2t + 3\sin 2t$$
$$\cos 2t(-4A + 4B + 5A) + \sin 2t(-4B - 4A + 5B) = 12\cos 2t + 3\sin 2t$$

$$A + 4B = 12 \xrightarrow{\times 4} 4A + 16B = 48$$
$$-4A + B = 3 \longrightarrow -4A + B = 3$$
$$17B = 51 \rightarrow B = 3$$

$$A + 4(3) = 12$$
$$A = 0$$

$$x = e^{-t}(C_1 \cos 2t + C_2 \sin 2t) + 3 \sin 2t$$

$$x(0) = 1(C_1 + 0) + 0 = 1 \rightarrow C_1 = 1$$

$$x' = e^{-t}(-2C_1 \sin 2t + 2C_2 \cos 2t) - e^{-t}(C_1 \cos 2t + C_2 \sin 2t) + 6 \cos 2t$$

$$x'(0) = 1(0 + 2C_2) - 1(1 + 0) + 6 = 5$$

$$2C_2 + 5 = 5$$

$$C_2 = 0$$

$$x = e^{-t} \cos 2t + 3 \sin 2t$$

y_c is called the transient term

y_p is called the steady–state term