Warm–up Problems

A force of 2 pounds stretches a spring 1 foot. A mass weighing 8 pounds is attached to the end. The system lies on a table that imparts a frictional force numerically equal to 1.5 times the instantaneous velocity. Initially, the mass is displaced <u>4 inches</u> above the equilibrium position and released from rest. Find the equation of motion.

Power Series Review

To solve linear DE's with variable coefficients, we'll be using power series

→ First, let's review

Power Series centered at $x = a$:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

→This is a function, and this may be as simple as the function gets

→For simplicity, we will usually use $a = 0$

$$1 - 3x + \frac{2}{5}x^2 - \frac{1}{17}x^3 - 7x^4 + \pi x^5 + \ldots$$

Some familiar functions can be represented as power series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

These are called <u>analytic functions</u> because they can be represented as power series, and are useful to remember.

A power series is convergent at a value of x if the infinite sum is equal to a finite number when evaluated at x.

The interval of convergence is the interval of x values that make the series converge. The radius of convergence is the distance away from $x = a$ that we can go to get convergence.

Radius = ∞ → Interval = $\mathbb{R}$

Radius = 0 → Interval = just the point a

The interval will be the domain of our function.

Ratio Test

Consider the series Σb_n

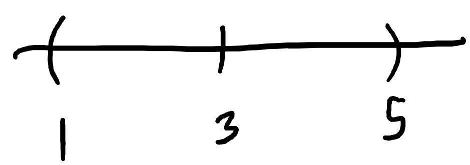
- converges ~~absolutely~~ if $\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| < 1$
- diverges if $\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| > 1$

Ex. Find the interval of convergence and radius of convergence for $\displaystyle\sum_{n=1}^{\infty}\frac{(x-3)^n}{n2^n}$

$$\lim_{n\to\infty}\left|\frac{\dfrac{(x-3)^{n+1}}{(n+1)\,2^{n+1}}}{\dfrac{(x-3)^n}{n\cdot 2^n}}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{\cancel{n+1}}}{(n+1)\,2^{\cancel{n+1}}}\cdot\frac{n\cdot\cancel{2^n}}{\cancel{(x-3)^n}}\right| = \lim_{n\to\infty}\left|\frac{x-3}{2}\cdot\underbrace{\frac{\cancel{n}}{\cancel{n+1}}}_{1}\right|$$

$$= \left|\frac{x-3}{2}\right| < 1$$

$$|x-3| < 2$$



I: $(1,5)$

$R = 2$

If $y = \sum_{n=0}^{\infty} c_n x^n$, then

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

We can add, subtract, multiply, and divide series term–by–term.

Ex. Express x^2e^x as a power series.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$x^2 e^x = x^2\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)$$

$$= x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \frac{x^6}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}$$

Ex. Express $e^x \sin x$ as a power series. (1st 4 nonzero terms)

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)$$

$$\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots\right)\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \dots\right)$$

$$x(1) + x^2(1) + x^3\left(-\frac{1}{6} + \frac{1}{2}\right) + x^4\left(-\frac{1}{6} + \frac{1}{6}\right) + x^5\left(\frac{1}{120} - \frac{1}{12} + \frac{1}{24}\right) + \dots$$

$$x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \dots$$

Ex. ~~Let~~ $y = \sum_{n=0}^{\infty} c_n x^n$, express $y'' + xy$ as a power series involving x^k.

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

$$\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + x \sum_{n=0}^{\infty} c_n x^n$$

$$\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

$k = n-2$
$n = k+2$

$k = n+1$
$n = k-1$

$$\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1) x^k + \sum_{k=1}^{\infty} c_{k-1} x^k$$

$$\underbrace{c_2 \cdot 2 \cdot 1 \cdot x^0}_{k=0} + \sum_{k=1} x^k \left[c_{k+2}(k+2)(k+1) + c_{k-1} \right]$$