Warm–up Problems

1. Find the interval of convergence and radius of convergence for $\sum \frac{n(x+2)^n}{3^{n+1}}$

2. If $y = \sum_{n=0}^{\infty} c_n x^n$, write $y'' - 4y$ as a single power series.

Power Series Solutions

Consider $y'' + P(x)y' + Q(x)y = 0$

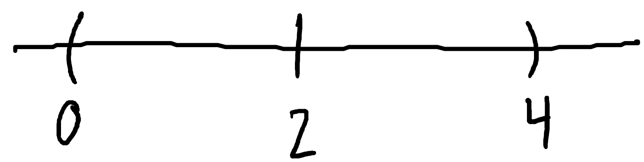
Def. x_0 is an ordinary point of the DE if $P(x)$ and $Q(x)$ are analytic at x_0 (can be written as a power series). x_0 is a singular point if it is not ordinary.

→ When we get to $y'' + P(x)y' + Q(x)y = f(x)$, all three functions must be analytic.

Thm. Existence of a Power Series Solution

If x_0 is an ordinary point of the DE, we can always find two linearly independent solutions that are power series centered at x_0. A series solution converges at least on some interval $|x - x_0| < R$, where R is the distance from x_0 to the nearest singular point.

Ex. For $y'' + \ln x \, y' + 3y = 0$, $x = 2$ is ordinary. Find the interval where a solution would converge.

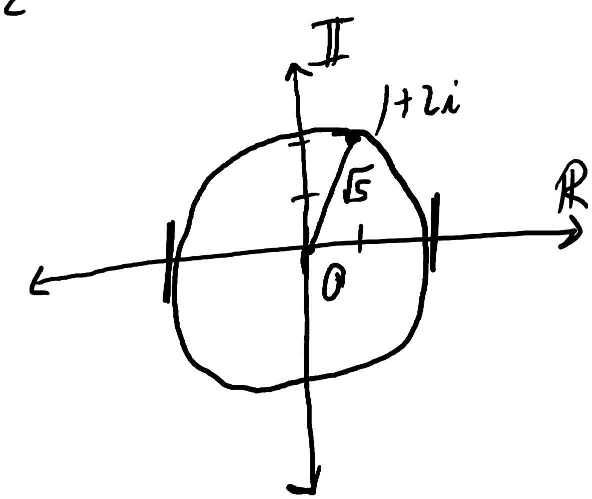


$(0, 4)$

Ex. For $(x^2 - 2x + 5)y'' + xy' - y = 0$, $x = 0$ is ordinary. Find the interval where a solution would converge.

$$y'' + \frac{x}{x^2-2x+5}y' - \frac{1}{x^2-2x+5}y = 0$$

$$x = \frac{2 \pm \sqrt{4 - 4 \cdot 5}}{2} = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$



$$\boxed{(-\sqrt{5}, \sqrt{5})}$$

→ The interval may be bigger, this just gives us a minimum.

Ex. Solve $y'' + xy = 0$.

$$2c_2 + \sum_{k=1}^{\infty}\left[(k+1)(k+2)c_{k+2} + c_{k-1}\right]x^k$$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

$$\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + x\sum_{n=0}^{\infty} c_n x^n = 0$$

$$2c_2 + \sum_{k=1}^{\infty}\left[(k+1)(k+2)c_{k+2} + c_{k-1}\right]x^k = 0$$

$$2c_2 = 0$$

$$c_2 = 0$$

$$(k+1)(k+2)c_{k+2} + c_{k-1} = 0$$

$$(k+1)(k+2)c_{k+2} = -c_{k-1} \rightarrow c_{k+2} = \frac{-1}{(k+1)(k+2)}c_{k-1}$$

$$c_0 = c_0$$

$$c_1 = c_1$$

$$c_2 = 0$$

$k=1 \rightarrow$ $c_3 = \frac{-1}{2\cdot 3}c_0 = -\frac{1}{6}c_0$

$k=2 \rightarrow$ $c_4 = \frac{-1}{3\cdot 4}c_1 = -\frac{1}{12}c_1$

$k=3 \rightarrow$ $c_5 = \frac{-1}{4\cdot 5}c_2 = 0$

$$c_6 = \frac{-1}{5\cdot 6}c_3 = \frac{-1}{30}\left(-\frac{1}{6}c_0\right) = \frac{1}{180}c_0$$

$$c_7 = \frac{-1}{6\cdot 7}c_4 = \frac{-1}{42}\left(\frac{-1}{12}c_1\right) = \frac{1}{504}c_1$$

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7 + \ldots$$

$$y = c_0 + c_1x + 0x^2 - \frac{1}{6}c_0x^3 - \frac{1}{12}c_1x^4 + 0x^5 + \frac{1}{180}c_0x^6 + \frac{1}{504}c_1x^7 + \ldots$$

$$y = c_0\underbrace{\left(1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 + \ldots\right)}_{y_1} + c_1\underbrace{\left(x - \frac{1}{12}x^4 + \frac{1}{504}x^7 + \ldots\right)}_{y_2}$$

Ex. Solve $(x^2+1)y''+xy'-y=0$.

$$(x^2+1)\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} c_n n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$x^2\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + 1\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} c_n n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} c_n n(n-1)x^n + \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} c_n n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$k=n$ $\quad k=n-2$, $n=k+2$ $\quad k=n$ $\quad k=n$

$$\sum_{k=2}^{\infty} c_k k(k-1)x^k + \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} c_k k x^k - \sum_{k=0}^{\infty} c_k x^k = 0$$

$$c_2\cdot 2\cdot 1\, x^0 + c_3\cdot 3\cdot 2\, x^1 + c_1\cdot 1\, x^1 - c_0 x^0 - c_1 x^1 + \sum_{k=2}^{\infty} x^k\left[c_k k(k-1) + c_{k+2}(k+2)(k+1) + c_k k - c_k\right] = 0$$

$k=0$ $\quad k=1$ $\quad k=1$ $\quad k=0$ $\quad k=1$ $\quad k=2$

$$(2c_2 - c_0) + x(6c_3 + c_1 - c_1)$$

$2c_2 - c_0 = 0$, $\quad 6c_3 + c_1 - c_1 = 0$

$c_2 = \frac{1}{2}c_0$ $\qquad c_3 = 0$

$[\ldots] = 0$

$$c_{k+2}(k+2)(k+1) = (-k(k-1) - k + 1)c_k$$

$$c_{k+2} = \frac{-k^2+1}{(k+2)(k+1)}c_k = \frac{1-k}{k+2}c_k$$

$C_2 = \frac{1}{2} C_0$ $\quad C_3 = 0$ $\quad C_{k+2} = \frac{1-k}{k+2} C_k$

$C_0 = C_0$

$C_1 = C_1$

$C_2 = \frac{1}{2} C_0$

$C_3 = 0$

$k=2 \rightarrow \quad C_4 = \frac{-1}{4} C_2 = \frac{-1}{4}\left(\frac{1}{2} C_0\right) = \frac{-1}{8} C_0$

$k=3 \rightarrow \quad C_5 = \frac{-2}{5} C_3 = -\frac{2}{5}(0) = 0$

$y = C_0 + C_1 x + \frac{1}{2} C_0 x^2 + 0 x^3 - \frac{1}{8} C_0 x^4 + 0 x^5 + \ldots$

$y = C_0\left(1 + \frac{1}{2} x^2 - \frac{1}{8} x^4 + \ldots\right) + C_1(x)$

Ex. When solving $y'' - (1+x)y = 0$, we obtain

$$c_2 = \tfrac{1}{2}c_0 \qquad c_{k+2} = \frac{c_k + c_{k-1}}{(k+1)(k+2)}$$

Find the two linearly independent solutions.

$c_0 = 1,\ c_1 = 0$

$c_0 = 1$

$c_1 = 0$

$c_2 = \frac{1}{2}c_0 = \frac{1}{2}$

$k=1 \to c_3 = \frac{c_1 + c_0}{2 \cdot 3} = \frac{1}{6}$

$y_1 = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots$

$c_0 = 0,\ c_1 = 1$

$c_0 = 0$

$c_1 = 1$

$c_2 = \frac{1}{2}c_0 = 0$

$k=1 \to c_3 = \frac{c_1 + c_0}{2 \cdot 3} = \frac{1}{6}$

$k=2 \to c_4 = \frac{c_2 + c_1}{3 \cdot 4} = \frac{1}{12}$

$y_2 = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \ldots$

$$y = D_1\left(1 + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \ldots\right) + D_2\left(x + \tfrac{1}{6}x^3 + \tfrac{1}{12}x^4 + \ldots\right)$$

<u>Ex.</u> Solve $(x^2 + 1)y'' + xy' - y = 0$, $y(0) = 2$, $y'(0) = 3$.

$$y = C_0\left(1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \dots\right) + C_1(x)$$

$$y(0) = C_0 = 2$$

$$y'(0) = C_1 = 3$$

$$y = 2\left(1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \dots\right) + 3x$$

Ex. Solve $y'' + (\cos x)y = 0$.

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} = 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \ldots$$

$$\left(2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \ldots\right) + \left(1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 + \ldots\right)\left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots\right) = 0$$

const: $2c_2 + c_0 = 0 \longrightarrow c_2 = -\frac{1}{2}c_0$

x : $6c_3 + c_1 = 0 \longrightarrow c_3 = -\frac{1}{6}c_1$

x^2: $12c_4 - \frac{1}{2}c_0 + c_2 = 0 \longrightarrow c_4 = \dfrac{\frac{1}{2}c_0 - c_2}{12} = \dfrac{\frac{1}{2}c_0 - \left(-\frac{1}{2}c_0\right)}{12} = \dfrac{c_0}{12}$

x^3: $20c_5 + c_3 - \frac{1}{2}c_1 = 0 \longrightarrow c_5 = \dfrac{\frac{1}{2}c_1 - c_3}{20} = \dfrac{\frac{1}{2}c_1 - \left(-\frac{1}{6}c_1\right)}{20} = \dfrac{\frac{2}{3}c_1}{20} = \dfrac{1}{30}c_1$

$$y = c_0 + c_1 x - \tfrac{1}{2}c_0 x^2 - \tfrac{1}{6}c_1 x^3 + \tfrac{1}{12}c_0 x^4 + \tfrac{1}{30}c_1 x^5 + \ldots$$

$$y = c_0\left(1 - \tfrac{1}{2}x^2 + \tfrac{1}{12}x^4 + \ldots\right) + c_1\left(x - \tfrac{1}{6}x^3 + \tfrac{1}{30}x^5 + \ldots\right)$$