Warm–up Problems

Find two power series solutions of the DE $y'' + xy' + y = 0$ about the ordinary point $x = 0$.

Solutions About Singular Points

x_0 is a singular point of $y'' + P(x)y' + Q(x)y = 0$ if $P(x)$ or $Q(x)$ are not analytic at x_0

→Usually this means we're dividing by 0.

<u>Def.</u> A singular point x_0 is <u>regular</u> if x_0 is a root at most once in the denominator of $P(x)$ and at most twice in the denominator of $Q(x)$.

$\underline{\text{Ex.}}$ $(x^2 - 4)^2 y'' + 3(x - 2)y' + 5y = 0$

$$y'' + \frac{3(x-2)}{(x^2-4)^2} y' + \frac{5}{(x^2-4)^2} y = 0$$

$$y'' + \frac{3}{(x-2)(x+2)^2} y' + \frac{5}{(x-2)^2(x+2)^2} y = 0$$

<u>singular</u>

$x = 2 \longrightarrow$ regular $\quad P(x) = \frac{\quad}{(x-2)} \qquad Q(x) = \frac{\quad}{(x-2)^2}$

$x = -2 \longrightarrow$ irregular $\quad P(x) = \frac{\quad}{(x+2)^2} \qquad Q(x) = \frac{\quad}{(x+2)^2}$

Thm. Frobenius' Theorem

If x_0 is a regular singular point, then there exists at least one solution of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

where r is some constant.

→Using this is called the Method of Frobenius

→The solution is an infinite series, but it may not be a power series.

Ex. $3xy'' + y' - y = 0$

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2}$$

$$3x\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2} + \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3c_n(n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$k = n-1$ $\quad$ $k = n-1$ $\quad$ $k = n$

$n = k+1$ $\quad$ $n = k+1$

$$\sum_{k=-1}^{\infty} 3c_{k+1}(k+r+1)(k+r)x^{k+r} + \sum_{k=-1}^{\infty} c_{k+1}(k+r+1)x^{k+r} - \sum_{k=0}^{\infty} c_k x^{k+r} = 0$$

$$3c_0 r(r-1)x^{r-1} + c_0 r x^{r-1} + \sum_{k=0}^{\infty} x^{k+r}\underbrace{\left[3c_{k+1}(k+r+1)(k+r) + c_{k+1}(k+r+1) - c_k\right]}_{=0} = 0$$

$$3c_0 r(r-1) + c_0 r = 0$$

$$c_0 r(3r-3+1) = 0$$

$$c_0 r(3r-2) = 0$$

$$c_0 \quad r = 0 \quad r = \frac{2}{3}$$

$$c_{k+1}\left[3(k+r+1)(k+r) + (k+r+1)\right] = c_k$$

$$c_{k+1} = \frac{1}{(k+r+1)(3k+3r+1)}c_k$$

$$C_{k+1} = \frac{1}{(k+r+1)(3k+3r+1)} C_k$$

$\underline{r=0}$: $C_{k+1} = \frac{1}{(k+1)(3k+1)} C_k$

$C_0 = 1$

$k=0 \to C_1 = \frac{1}{1 \cdot 1} \cdot C_0 = 1$

$k=1 \to C_2 = \frac{1}{2 \cdot 4} C_1 = \frac{1}{8}$

$$y_1 = \left(1 + 1x + \frac{1}{8}x^2 + \dots\right) x^0$$

$\underline{r = \frac{2}{3}}$: $C_{k+1} = \frac{1}{(k+\frac{2}{3}+1)(3k+2+1)} C_k$

$= \frac{1}{(k+\frac{5}{3})(3k+3)} C_k$

$= \frac{1}{(3k+5)(k+1)} C_k$

$C_0 = 1$

$k=0 \to C_1 = \frac{1}{5 \cdot 1} C_0 = \frac{1}{5}$

$k=1 \to C_2 = \frac{1}{8 \cdot 2} C_1 = \frac{1}{16} \cdot \frac{1}{5} = \frac{1}{80}$

$$y_2 = \left(1 + \frac{1}{5}x + \frac{1}{80}x^2 + \dots\right) x^{2/3}$$

→ $r(3r - 2) = 0$ is called the indical equation

→ $r = 0$ and $r = 2/3$ are called the indical roots

Note that $xP(x)$ and $x^2Q(x)$ are analytic, so

$$xP(x) = \underline{a_0} + a_1x + \ldots$$

$$x^2Q(x) = \underline{b_0} + b_1x + \ldots$$

→ The indical equation is $r(r - 1) + a_0r + b_0 = 0$

<u>Ex.</u> Find the indical roots of $2xy'' + (1 + x)y' + y = 0$

$$P(x) = \frac{1+x}{2x} \longrightarrow xP(x) = \frac{1+x}{2} = \frac{1}{2} + \frac{1}{2}x$$

$$Q(x) = \frac{1}{2x} \longrightarrow x^2 Q(x) = \frac{x}{2} = 0 + \frac{1}{2}x$$

$$r(r-1) + \frac{1}{2}r + 0 = 0$$

$$r^2 - \frac{1}{2}r = 0$$

$$r\left(r - \frac{1}{2}\right) = 0$$

$$r = 0, \frac{1}{2}$$

Ex. $xy'' + y = 0$

$$x\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$k = n-1$ $\qquad\qquad$ $k = n$

$n = k+1$

$$\sum_{k=-1}^{\infty} c_{k+1}(k+r+1)(k+r)x^{k+r} + \sum_{k=0}^{\infty} c_k x^{k+r} = 0$$

$$c_0 r(r-1)x^{r-1} + \sum_{k=0}^{\infty} x^{k+r}\Big[\underbrace{c_{k+1}(k+r+1)(k+r) + c_k}_{=0}\Big] = 0$$

$r(r-1) = 0$

$r = 0,\ r = 1$

$$c_{k+1}(k+r+1)(k+r) = -c_k$$

$$c_{k+1} = \frac{-1}{(k+r+1)(k+r)}c_k$$

$$c_{k+1} = \frac{-1}{(k+r+1)(k+r)} c_k$$

$\underline{r=1:}$ $c_{k+1} = \frac{-1}{(k+2)(k+1)} c_k$

$c_0 = 1$

$k=0 \to c_1 = \frac{-1}{2\cdot 1} c_0 = -\frac{1}{2}$

$k=1 \to c_2 = \frac{-1}{3\cdot 2} c_1 = \frac{1}{12}$

$y_1 = \left(1 - \frac{1}{2}x + \frac{1}{12}x^2 + \ldots\right)x^1$

$= x - \frac{1}{2}x^2 + \frac{1}{12}x^3 + \ldots$

$\underline{r=0:}$ $c_{k+1} = \frac{-1}{(k+1)k} c_k$

$c_1 = 1$

$k=1 \to c_2 = \frac{-1}{2\cdot 1} c_1 = -\frac{1}{2}$

$k=2 \to c_3 = \frac{-1}{3\cdot 2} c_2 = \frac{1}{12}$

$y_2 = \left(1x - \frac{1}{2}x^2 + \frac{1}{12}x^3 + \ldots\right)x^0$

$= x - \frac{1}{2}x^2 + \frac{1}{12}x^3 + \ldots$

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The values of r determine our solutions:

Case 1: $r_1 - r_2$ is not an integer

→Frobenius will give us two solutions

Case 2: $r_1 - r_2$ is an integer or $r_1 = r_2$

→We may still get two solutions from Frobenius

→If not, we use reduction of order to find the second solution

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2} \longrightarrow y'' + P(x)y' + Q(x)y = 0$$

Ex. Find the second solution to $xy'' + y = 0$

$$y_1 = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 + \dots.$$

$$y_2 = y_1 \int \frac{e^{-\int 0\,dx}}{\left(x - \frac{1}{2}x^2 + \frac{1}{12}x^3 + \dots\right)^2}\,dx = \left(x - \frac{1}{2}x^2 + \frac{1}{12}x^3 + \dots\right) \int \frac{1}{\left(x - \frac{1}{2}x^2 + \frac{1}{12}x^3 + \dots\right)^2}\,dx$$