Laplace Transform

A transform is an operation that changes a function into a new function.

→ Examples of this are derivatives and antiderivatives.

→ These are linear transforms:

$$\int \left(\alpha f(x) + \beta g(x)\right) dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$$

Def. Let $f(t)$ be defined for $t \geq 0$, then

$$\mathcal{L}\left\{f\left(t\right)\right\} = \int_0^\infty e^{-st} f\left(t\right) dt$$

is called the Laplace Transform of f.

*** $\mathcal{L}\{f(t)\} = F(s)$

Ex. Find $\mathcal{L}\{1\}$. $= \int_0^\infty e^{-st}\cdot 1\,dt$

$$= \lim_{R\to\infty} \int_0^R e^{-st}\,dt = \lim_{R\to\infty} \frac{1}{-s}e^{-st}\Big|_0^R$$

$$= \lim_{R\to\infty} \frac{1}{-s}e^{-sR} + \frac{1}{s}e^{-s\cdot 0} = \lim_{R\to\infty} \underbrace{\cancel{\frac{-1}{se^{sR}}}}_{0} + \frac{1}{s} = \frac{1}{s}$$

$$\boxed{s>0}$$

Find the domain.

Ex. Find $\mathcal{L}\{t\}$. $= \int_0^\infty e^{-st} \cdot t \, dt$

$$\boxed{\begin{array}{ll} u = t & dv = e^{-st} dt \\ du = dt & v = \frac{1}{-s} e^{-st} \end{array}}$$

$$= \frac{t}{-s} e^{-st} - \int_0^\infty \frac{1}{-s} e^{-st} dt$$

$$= \frac{t}{-s} e^{-st} + \frac{1}{s} \cdot \frac{1}{-s} e^{-st} \Big|_0^\infty$$

$$= \frac{t}{-s e^{st}} - \frac{1}{s^2 e^{st}} \Big|_0^\infty$$

$$= \left(\cancel{\frac{\infty}{-s e^{s\infty}}}^{\to 0} - \cancel{\frac{1}{s^2 e^{s \cdot \infty}}}^{\to 0} \right) - \left(\cancel{\frac{0}{-s e^{s \cdot 0}}}^{\to 0} - \frac{1}{s^2 e^{s \cdot 0}} \right)$$

$$= \frac{1}{s^2} \qquad s > 0$$

<u>Ex.</u> Find $\mathcal{L}\{e^{-3t}\}$. $= \int_0^\infty e^{-st} \cdot e^{-3t}\, dt = \int_0^\infty e^{-t(s+3)}\, dt$

$$= \frac{1}{-(s+3)} e^{-t(s+3)} \Big|_0^\infty$$

$$= \underbrace{\frac{1}{-(s+3)e^{\infty(s+3)}}}_{0} - \frac{1}{-(s+3)e^{0(s+3)}}$$

$$= \frac{1}{s+3}, \quad s > -3$$

Some basic transforms are on p. 277

→ $\mathcal{L}$ is linear

<u>Ex.</u> Find $\mathcal{L}\{1 + 5t\}$. $= \mathcal{L}\{1\} + \mathcal{L}\{5t\}$

$$= \mathcal{L}\{1\} + 5\mathcal{L}\{t\}$$

$$= \frac{1}{s} + \frac{5}{s^2}$$

A function f is of <u>exponential order</u> c if there exist constants c, $M > 0$, and $T > 0$ such that $|f(t)| < Me^{ct}$ for all $t > T$.

<u>Thm.</u> If f is a piecewise continuous function on $[0,\infty)$ and of exponential order c, then $\mathcal{L}\{f(t)\}$ exists for $s > c$.

→ This is not a requirement for the existence of $\mathcal{L}$, others also have an $\mathcal{L}$.

Thm. If f is piecewise continuous on $(0,\infty)$ and of exponential order, and if $F(s) = \mathcal{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$

$F(s) = s + 2$

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Ex. Find $\mathcal{L}\{f(t)\}$, where $f(t)=\begin{cases}0 & 0\le t\le 3\\ 2 & t>3\end{cases}$

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^3 \cancel{e^{-st}\cdot 0}\,dt + \int_3^\infty e^{-st}\cdot 2\,dt$$

$$\cancel{e^{-st}\cdot 0} \to 0$$

$$= \frac{2}{-s} e^{-st}\Big|_3^\infty = \cancel{\frac{2}{-s e^{s\cdot\infty}}} - \frac{2}{-s e^{s\cdot 3}} = \boxed{\frac{2}{s e^{3s}}}$$

$$\cancel{\frac{2}{-s e^{s\cdot\infty}}} \to 0$$