# Inverse Transforms and Derivatives

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}=1 \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}=t \qquad \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}=e^{-3t}$$

$$\mathcal{L}^{-1}\left\{F(s)\right\}=f(t)$$

See p. 282 for some inverse transforms

Ex. Find $\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{24}\mathcal{L}^{-1}\left\{\frac{4!}{s^{4+1}}\right\}$

$$= \frac{1}{24}t^4$$

$$\mathcal{L}^{-1}\left\{\frac{1}{24}\left(\frac{4!}{s^{4+1}}\right)\right\}$$

**THEOREM 7.2.1 Some Inverse Transforms**

(a) $1 = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\}$

(b) $t^n = \mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}, \quad n = 1, 2, 3, \ldots$

(c) $e^{at} = \mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\}$

(d) $\sin kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}$

(e) $\cos kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$

(f) $\sinh kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\}$

(g) $\cosh kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$

**Ex.** Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+7}\right\} = \dfrac{1}{\sqrt{7}}\mathcal{L}^{-1}\left\{\dfrac{\sqrt{7}}{s^2+(\sqrt{7})^2}\right\}$

$= \dfrac{1}{\sqrt{7}}\sin(\sqrt{7}\,t)$

$\dfrac{1}{3s^2+12}$

$\dfrac{1}{3}\,\dfrac{1}{s^2+4}$

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**Ex.** Find $\mathcal{L}^{-1}\left\{\dfrac{-2s+6}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\dfrac{-2s}{s^2+4} + \dfrac{6}{s^2+4}\right\}$

$$= -2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= -2\cos 2t + 3\sin 2t$$

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<u>Ex.</u> Find $\mathcal{L}^{-1}\left\{\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{-16/5}{s-1} + \dfrac{25/6}{s-2} + \dfrac{1/30}{s+4}\right\}$

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$

$$s^2+6s+9 = A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)$$

$$s=1: \quad 1+6+9 = A(-1)(5) \longrightarrow A = -\frac{16}{5}$$

$$s=2: \quad 4+12+9 = B(1)(6) \rightarrow B = \frac{25}{6}$$

$$s=-4: \quad 16-24+9 = C(-5)(-6) \rightarrow C = \frac{1}{30}$$

$$= -\frac{16}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

Ex. Find $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$

$$\boxed{\mathcal{L}\{f(t)\} = F(s)}$$

$$\boxed{\begin{array}{ll} u = e^{-st} & dv = f'(t)\,dt \\ du = -se^{-st}\,dt & v = f(t) \end{array}}$$

$$= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty -s\,e^{-st} f(t)\,dt$$

$$= \cancel{\frac{f(\infty)}{e^{s\cdot\infty}}}^{\,0} - \frac{f(0)}{e^{s\cdot 0}} + s\underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{F(s)}$$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Ex. Find $\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt$

$$\boxed{\begin{array}{ll} u = e^{-st} & dv = f''(t)\,dt \\ du = -se^{-st}\,dt & v = f'(t) \end{array}}$$

$$= e^{-st} f'(t)\Big|_0^\infty - \int_0^\infty -se^{-st} f'(t)\,dt$$

$$= \underbrace{\cancel{\frac{f'(\infty)}{e^{s\cdot\infty}}}}_{0} - \frac{f'(0)}{e^{s\cdot 0}} + s\underbrace{\int_0^\infty e^{-st} f'(t)\,dt}_{\mathcal{L}\{f'\}}$$

$$= s\left[sF(s) - f(0)\right] - f'(0)$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

In general,

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$$

→ We can use this to solve an IVP

Ex. Solve $\mathcal{L}\{y' + 3y = 13\sin 2t\}$; $y(0) = 6$.

$$\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = 13\,\mathcal{L}\{\sin 2t\}$$

$$sF(s) - f(0) + 3F(s) = 13\frac{2}{s^2+4}$$

$$sF(s) - 6 + 3F(s) = \frac{26}{s^2+4}$$

$$sF(s) + 3F(s) = \frac{26}{s^2+4} + \frac{6(s^2+4)}{s^2+4}$$

$$F(s)(s+3) = \frac{6s^2+50}{s^2+4}$$

$$F(s) = \frac{6s^2+50}{(s^2+4)(s+3)}$$

$$y = \mathcal{L}^{-1}\left\{\frac{6s^2+50}{(s^2+4)(s+3)}\right\}$$

$$y = \mathcal{L}^{-1}\left\{\frac{-2s+6}{s^2+4} + \frac{8}{s+3}\right\}$$

$$-2\cos 2t + 3\sin 2t + 8e^{-3t}$$

| $y = f(t)$ |
| --- |
| $\mathcal{L}\{y\} = F(s)$ |

$$\frac{6s^2+50}{(s^2+4)(s+3)} = \frac{As+B}{s^2+4} + \frac{C}{s+3}$$

$$6s^2+50 = (As+B)(s+3) + C(s^2+4)$$

$s=-3 \rightarrow 6\cdot 9+50 = C(13) \rightarrow C = \frac{104}{13} = 8$

$s=0 \rightarrow 50 = 3B+4C \rightarrow 50 = 3B+32$

$3B = 18$

$B = 6$

$s=1 \rightarrow 56 = (A+6)(4) + 8(5)$

$56 = 4A + 24 + 40$

$4A = -8 \rightarrow A = -2$

Ex. Solve $\mathcal{L}\{y'' - 3y' + 2y = e^{-4t}\}$, $y(0) = 1$, $y'(0) = 5$.

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-4t}\}$$

$$\left[s^2F(s) - sf(0) - f'(0)\right] - 3\left[sF(s) - f(0)\right] + 2F(s) = \frac{1}{s+4}$$

$$\left[s^2F(s) - s - 5\right] - 3\left[sF(s) - 1\right] + 2F(s) = \frac{1}{s+4}$$

$$s^2F(s) - \cancel{s} - 5 - 3sF(s) + 3 + 2F(s) = \frac{1}{s+4}$$

$$s^2F(s) - 3sF(s) + 2F(s) = \frac{1}{s+4} + \frac{(s+2)(s+4)}{s+4}$$

$$F(s)(s^2 - 3s + 2) = \frac{s^2+6s+9}{s+4}$$

$$F(s) = \frac{s^2+6s+9}{(s+4)(s-1)(s-2)}$$

$$y = \mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} = \frac{-16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

We had other methods to solve both of these problems, but note that the initial values and non-homogeneous parts are built in…

→ No worrying about cases, solving for constants, $y_c$ and $y_p$, or variation of parameters.