Linear Systems

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + a_{13}(t)x_3 + f_1(t)$$
$$\frac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2 + a_{23}(t)x_3 + f_2(t)$$
$$\frac{dx_3}{dt} = a_{31}(t)x_1 + a_{32}(t)x_2 + a_{33}(t)x_3 + f_3(t)$$

If $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$

then the system can be written $X' = \boxed{AX} + F$

→ If $F = 0$, the system is homogeneous

→ Finding a solution means finding X.

Ex. Write the system in matrix form.

$$x' = 6x + y + z + t$$

$$y' = 8x + 7y - z + 10t$$

$$z' = 2x + 9y - z + 6t$$

$$X' = \begin{pmatrix} 6 & 1 & 1 \\ 8 & 7 & -1 \\ 2 & 9 & -1 \end{pmatrix} X + \begin{pmatrix} t \\ 10t \\ 6t \end{pmatrix}$$

Ex. Verify that $X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$ and $X_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$ are

solutions to $X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X$

$$X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}$$

$$X_1' = \begin{pmatrix} -2e^{-2t} \\ 2e^{-2t} \end{pmatrix}$$

$$\begin{pmatrix} -2e^{-2t} \\ 2e^{-2t} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-2t} - 3e^{-2t} \\ 5e^{-2t} - 3e^{-2t} \end{pmatrix} = \begin{pmatrix} -2e^{-2t} \\ 2e^{-2t} \end{pmatrix}$$

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$$X_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t} \rightarrow X_2' = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \cdot 6e^{6t}$$

$$\underbrace{\begin{pmatrix} 18 \\ 30 \end{pmatrix} e^{6t}}_{X'} = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \underbrace{\begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}}_{X}$$

$$= \begin{pmatrix} 3 + 15 \\ 15 + 15 \end{pmatrix} e^{6t}$$

$$= \begin{pmatrix} 18 \\ 30 \end{pmatrix} e^{6t}$$

✓

An IVP means that we are given $X\left(t_0\right)=\begin{pmatrix} x_1\left(t_0\right) \\ x_2\left(t_0\right) \\ x_3\left(t_0\right) \end{pmatrix}$

Superposition still applies

→ If $X_1, X_2, \ldots, X_n$ are solutions, then so is

$$X = C_1X_1 + C_2X_2 + \ldots + C_nX_n$$

Linear independence still applies

→ $X_1, X_2, \ldots, X_n$ are linearly independent if there are no non-zero coefficients such that

$$C_1X_1 + C_2X_2 + \ldots + C_nX_n = 0$$

Thm. $X_1, X_2, \ldots, X_n$ are linearly independent iff

$$W(X_1, X_2, \ldots, X_n) = \begin{array}{c} \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} \\ \begin{array}{cccc} X_1 & X_2 & \quad & X_n \end{array} \end{array} \neq 0$$

<u>Ex.</u> Show that $X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$ and $X_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$ are linearly independent.

$$X_1 \downarrow \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix} \qquad X_2 \downarrow \begin{pmatrix} 3e^{6t} \\ 5e^{6t} \end{pmatrix}$$

$$W = \begin{vmatrix} e^{-2t} & 3e^{6t} \\ -e^{-2t} & 5e^{6t} \end{vmatrix} = 5e^{4t} - \left(-3e^{4t}\right) = 8e^{4t} \neq 0$$

$\therefore$ indep.

Any set of n linearly independent solutions of the linear system of n equations is a fundamental set, and their linear combination is the general solution to the system.

→ $X = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$ is the general solution to

$$X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X$$

For non-homogeneous systems:

X_c = solution to homogeneous = $C_1X_1 + C_2X_2 + \ldots + C_nX_n$

X_p = particular solution to non-homogenous

$X = X_c + X_p$ is the general solution to non-homogenous

$\underline{\text{Ex.}}$ $X_p = \begin{pmatrix} 3t-4 \\ -5t+6 \end{pmatrix}$ is a solution to $\underbrace{X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X}$ $+ \begin{pmatrix} 12t-11 \\ -3 \end{pmatrix}$

$$X_c = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t}$$

$$\boxed{X = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t} + \begin{pmatrix} 3t-4 \\ -5t+6 \end{pmatrix}}$$