Homogeneous Linear Systems

<u>Thm.</u> Let λ_1, λ_2, λ_3 be distinct eigenvalues of A with corresponding eigenvectors K_1, K_2, K_3. Then the general solution to $X' = AX$ is

$$X = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t} + c_3 K_3 e^{\lambda_3 t}$$

<u>Ex.</u> Solve $\begin{array}{l}\frac{dx}{dt} = 2x + 3y \\ \frac{dy}{dt} = 2x + y\end{array}$, $X(0) = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ $\quad X' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} X$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 6 = \lambda^2 - 3\lambda - 4$$

$$= (\lambda - 4)(\lambda + 1) = 0$$

$$\lambda = 4, -1$$

$\underline{\lambda_1 = 4}$: $(A - 4I)K = 0$

$$\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$2a - 3b = 0$$

$$a = \frac{3}{2}b$$

$$K_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$\underline{\lambda_2 = -1}$ $(A - (-1)I)K = 0$

$$\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$2a + 2b = 0$$

$$b = -a$$

$$K_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$X = c_1\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{4t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t}$$

$$X(0) = c_1\begin{pmatrix} 3 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$$3c_1 + c_2 = 5$$

$$2c_1 - c_2 = 0 \rightarrow 2 - c_2 = 0$$

$$5c_1 = 5 \qquad c_2 = 2$$

$$c_1 = 1$$

$$X = \begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{4t} + 2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t}$$

Ex. Solve
$$\frac{dx}{dt} = -4x + y + z$$
$$\frac{dy}{dt} = x + 5y - z \longrightarrow X' = \begin{pmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{pmatrix} X$$
$$\frac{dz}{dt} = y - 3z$$

$$\det(A-\lambda I) = \begin{vmatrix} -4-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 0 & 1 & -3-\lambda \end{vmatrix} = (-4-\lambda)\begin{vmatrix} 5-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & -3-\lambda \end{vmatrix} + 0\begin{vmatrix} \sim & \sim \\ - & \sim \end{vmatrix}$$

$$= (-4-\lambda)\left[(5-\lambda)(-3-\lambda)+1\right] - (-3-\lambda-1)$$

$$= (-4-\lambda)\left[(5-\lambda)(-3-\lambda)+1\right] - (-4-\lambda)$$

$$= (-4-\lambda)\left[(5-\lambda)(-3-\lambda)+1 - 1\right] = (-4-\lambda)(5-\lambda)(-3-\lambda) = 0$$

$$\lambda_1 = -4 \qquad \lambda_2 = 5 \qquad \lambda_3 = -3$$

$\underline{\lambda_1 = -4:}$ $(A + 4I)K = 0$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 9 & -1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$b + c = 0 \rightarrow c = -b$

$a + 9b - c = 0$

$a + 9b + b = 0$

$a = -10b$

$$K_1 = \begin{pmatrix} -10 \\ 1 \\ -1 \end{pmatrix}$$

$\underline{\lambda_2 = 5:}$ $(A - 5I)K = 0$

$$\begin{pmatrix} -9 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -8 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$a - c = 0 \longrightarrow a = c$

$b - 8c = 0 \rightarrow b = 8c$

$$K_2 = \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix}$$

$\underline{\lambda_3 = -3:}$ $(A + 3I)K = 0$

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & 8 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$-a + b + c = 0$

$b = 0$

$-a + c = 0$

$a = c$

$$K_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$X = c_1\begin{pmatrix} -10 \\ 1 \\ -1 \end{pmatrix}e^{-4t} + c_2\begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix}e^{5t} + c_3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}e^{-3t}$$

When eigenvalues are not distinct, things change

→If λ_1 repeats twice and has two eigenvectors, the solution is

$$X = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_1 t}$$

Ex. Solve $X' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} X$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} - (-2)\begin{vmatrix} -2 & -2 \\ 2 & 1-\lambda \end{vmatrix} + 2\begin{vmatrix} -2 & 1-\lambda \\ 2 & -2 \end{vmatrix}$$

$$= (1-\lambda)\left[(1-\lambda)^2-4\right] + 2\left[-2(1-\lambda)+4\right] + 2\left[4-2(1-\lambda)\right]$$

$$= (1-\lambda)\left[(1-\lambda)^2-4\right] + 2(2\lambda+2) + 2(2\lambda+2)$$

$$= (1-\lambda)(1-\lambda+2)(1-\lambda-2) + 8(\lambda+1) = (\lambda+1)\left[-(1-\lambda)(3-\lambda) + 8\right]$$

$$= (\lambda+1)(-\lambda^2+4\lambda+5) = -(\lambda+1)(\lambda^2-4\lambda-5)$$

$$= -(\lambda+1)^2(\lambda-5) = 0$$

$\lambda_1 = 5 \qquad \lambda_2 = -1$

$\underline{\lambda_1 = 5}$: $(A - 5I)K = 0$

$$\begin{pmatrix} -4 & -2 & 2 \\ -2 & -4 & -2 \\ 2 & -2 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$-4a - 2b + 2c = 0$$
$$-2a - 4b - 2c = 0$$
$$-6a - 6b = 0 \to b = -a$$

$$-2a + 4a - 2c = 0$$
$$2a - 2c = 0$$
$$a = c$$

$$K_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$\underline{\lambda_2 = -1}$: $(A + I)K = 0$

$$\begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$2a - 2b + 2c = 0$$
$$b = a + c$$

$$K_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad K_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$X = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5t} + C_2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{-t} + C_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t}$$

If λ repeats twice and has eigenvector K, the solutions are $X_1 = Ke^{\lambda t}$ and $X_2 = Kte^{\lambda t} + Pe^{\lambda t}$ where P is given by

$$\left(A - \lambda I\right)P = K$$

Ex. Solve $X' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} X$

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix} = (3-\lambda)(-9-\lambda)+36 = \lambda^2+6\lambda-27+36 = \lambda^2+6\lambda+9$$
$$= (\lambda+3)^2 = 0 \to \lambda = -3$$

$\lambda = -3$: $(A+3I)K = 0$

$$\begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$2a - 6b = 0$

$a = 3b$

$$K = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$(A+3I)P = K$

$$\begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$2p - 6q = 1$

$$P = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

$$X = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t} + c_2\left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^{-3t}\right]$$

If λ repeats three times and has eigenvector K, the solutions are $X_1 = Ke^{\lambda t}$, $X_2 = Kte^{\lambda t} + Pe^{\lambda t}$, and $X_3 = K\frac{t^2}{2}e^{\lambda t} + Pte^{\lambda t} + Qe^{\lambda t}$, where Q is given by

$$\left(A - \lambda I\right)Q = P$$