Complex Eigenvalues and Non-Homogeneous Systems

If $\lambda = \alpha + \beta i$ is an eigenvalue, then so is $\bar{\lambda} = \alpha - \beta i$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} i \longrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} i$

If $\lambda = 5 + 2i$, $K = \begin{pmatrix} 1 \\ 1 - 2i \end{pmatrix}$, then

$$\bar{\lambda} = 5 - 2i, \; \bar{K} = \begin{pmatrix} 1 \\ 1 + 2i \end{pmatrix}$$

Define $B_1 = \text{Re}(K)$ and $B_2 = \text{Im}(K)$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad\qquad$ $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

<u>Thm.</u> Let $\lambda = \alpha + \beta i$ be an eigenvalue of A in the linear system $X' = AX$. Then the solutions of the system are

$$X_1 = [B_1 \cos \beta t - B_2 \sin \beta t]e^{\alpha t}$$

$$X_2 = [B_2 \cos \beta t + B_1 \sin \beta t]e^{\alpha t}$$

<u>Ex.</u> Solve $X' = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix} X, \; X(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 8 \\ -1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda)+8 = \lambda^2-4+8 = \lambda^2+4 = 0$$

$$\lambda = 0 \pm 2i$$

$\underline{\lambda = 2i}$: $(A-2iI)K = 0$

$$\begin{pmatrix} 2-2i & 8 \\ -1 & -2-2i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$-a + (-2-2i)b = 0$$

$$a = (-2-2i)b$$

$$K = \begin{pmatrix} -2-2i \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} -2 \\ 1 \end{pmatrix}}_{B_1} + \underbrace{\begin{pmatrix} -2 \\ 0 \end{pmatrix}}_{B_2} i$$

$$X = C_1\left[\begin{pmatrix} -2 \\ 1 \end{pmatrix}\cos 2t - \begin{pmatrix} -2 \\ 0 \end{pmatrix}\sin 2t\right] + C_2\left[\begin{pmatrix} -2 \\ 0 \end{pmatrix}\cos 2t + \begin{pmatrix} -2 \\ 1 \end{pmatrix}\sin 2t\right]$$

$$X(0) = C_1\begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2\begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \rightarrow \begin{matrix} -2C_1 - 2C_2 = 3 \\ C_1 = 5 \end{matrix}$$

$$-10 - 2C_2 = 3$$

$$-2C_2 = 13$$

$$C_2 = \frac{-13}{2}$$

$$X = 5\left[\begin{pmatrix} -2 \\ 1 \end{pmatrix}\cos 2t - \begin{pmatrix} -2 \\ 0 \end{pmatrix}\sin 2t\right] - \frac{13}{2}\left[\begin{pmatrix} -2 \\ 0 \end{pmatrix}\cos 2t + \begin{pmatrix} -2 \\ 1 \end{pmatrix}\sin 2t\right]$$

Non-homogeneous systems look like $X' = AX + F$ and the solution is $X = X_c + X_p$

To find X_p, we can use undetermined coefficients or variation of parameters

Ex. Solve $X' = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} X + \begin{pmatrix} -8 \\ 3 \end{pmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = (-1-\lambda)(1-\lambda) + 2 = \lambda^2 - 1 + 2 = \lambda^2 + 1 = 0 \rightarrow \lambda = 0 \pm 1i$$

$\underline{\lambda = i}$: $(A - iI)K = 0$

$$\begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$-a + (1-i)b = 0$$

$$a = (1-i)b$$

$$K = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} i$$

$$X_p = \begin{pmatrix} E \\ F \end{pmatrix} \rightarrow X_p' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} E \\ F \end{pmatrix} + \begin{pmatrix} -8 \\ 3 \end{pmatrix} \Rightarrow \begin{array}{l} 0 = -E + 2F - 8 \\ -\ 0 = -E + F + 3 \\ \hline 0 = F - 11 \rightarrow F = 11 \end{array}$$

$$0 = -E + 22 - 8$$

$$E = 14$$

$$X = C_1\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin t\right] + C_2\left[\begin{pmatrix} -1 \\ 0 \end{pmatrix}\cos t + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin t\right] + \begin{pmatrix} 14 \\ 11 \end{pmatrix}$$

$\underline{\text{Ex.}}$ Solve $\frac{dx}{dt} = 6x + y + 6t$

$\frac{dy}{dt} = 4x + 3y - 10t + 4$ $\Rightarrow X' = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix} X + \begin{pmatrix} 6t \\ -10t+4 \end{pmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & 1 \\ 4 & 3-\lambda \end{vmatrix} = (6-\lambda)(3-\lambda) - 4 = \lambda^2 - 9\lambda + 14 = (\lambda-2)(\lambda-7) = 0$$

$\lambda = 2 \quad \lambda = 7$

$\underline{\lambda = 2}$: $(A - 2I)K = 0$

$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$4a + b = 0$

$b = -4a$

$$K_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$\underline{\lambda = 7}$: $(A - 7I)K = 0$

$$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$-a + b = 0$

$a = b$

$$K_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$X_p = \begin{pmatrix} Pt+Q \\ Rt+S \end{pmatrix} \rightarrow X_p' = \begin{pmatrix} P \\ R \end{pmatrix}$$

$$\begin{pmatrix} P \\ R \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} Pt+Q \\ Rt+S \end{pmatrix} + \begin{pmatrix} 6t \\ -10t+4 \end{pmatrix} \Rightarrow \begin{cases} P = 6(Pt+Q)+1(Rt+S)+6t \\ R = 4(Pt+Q)+3(Rt+S)-10t+4 \end{cases}$$

$$\Rightarrow \begin{cases} P = t(6P+R+6)+(6Q+S) \\ R = t(4P+3R-10)+(4Q+3S+4) \end{cases}$$

$$6P+R+6=0 \xrightarrow{\times -3} -18P-3R-18=0$$
$$4P+3R-10=0 \rightarrow 4P+3R-10=0$$
$$-14P-28=0$$
$$P=-2$$
$$-12+R+6=0$$
$$R=6$$

$$6Q+S=\cancel{R}\,-2 \xrightarrow{\times -3} -18Q-3S=6$$
$$4Q+3S+4=\cancel{R}\,6 \rightarrow 4Q+3S+4=6$$
$$-14Q+4=12$$
$$-14Q=8$$
$$Q=\frac{-4}{7}$$
$$-\frac{24}{7}+S=-2$$
$$S=\frac{10}{7}$$

$$\boxed{X = c_1\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{7t} + \begin{pmatrix} -2t-4/7 \\ 6t+10/7 \end{pmatrix}}$$

Variation of Parameters

Let $X_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix}, X_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix}, X_3 = \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix}$ be the solutions to the homogeneous system, define

$$\varphi(t) = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$ ← Fundamental Matrix

The solution to the non-homogeneous system is

$$X = \underbrace{\varphi(t)\overrightarrow{C}}_{X_c} + \underbrace{\varphi(t)\int \varphi^{-1}(t)F(t)\,dt}_{X_p}$$

<u>Ex.</u> Solve $X' = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix} X + \begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}$

$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 1 \\ 2 & -4-\lambda \end{vmatrix} = (-3-\lambda)(-4-\lambda) - 2 = \lambda^2 + 7\lambda + 10 = (\lambda+5)(\lambda+2) = 0$

$\lambda = -5, -2$

$\underline{\lambda = -5:}$ $(A + 5I)K = 0$

$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$

$2a + b = 0$

$b = -2a$

$K_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$X_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-5t} = \begin{pmatrix} e^{-5t} \\ -2e^{-5t} \end{pmatrix}$

$\underline{\lambda = -2:}$ $(A + 2I)K = 0$

$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$

$-a + b = 0$

$b = a$

$K_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} = \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix}$

$\varphi = \begin{pmatrix} e^{-5t} & e^{-2t} \\ -2e^{-5t} & e^{-2t} \end{pmatrix}$

$\varphi^{-1} = \dfrac{1}{e^{-7t} + 2e^{-7t}} \begin{pmatrix} e^{-2t} & -e^{-2t} \\ 2e^{-5t} & e^{-5t} \end{pmatrix}$

$= \dfrac{1}{3} \begin{pmatrix} e^{5t} & -e^{5t} \\ 2e^{2t} & e^{2t} \end{pmatrix}$

$$\varphi^{-1}\cdot\vec{F} = \frac{1}{3}\begin{pmatrix} e^{5t} & -e^{5t} \\ 2e^{2t} & e^{2t}\end{pmatrix}\begin{pmatrix} 3t \\ e^{-t}\end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3te^{5t} - e^{4t} \\ 6te^{2t} + e^{t}\end{pmatrix}$$

$$\int \varphi^{-1}\cdot\vec{F} = \frac{1}{3}\begin{pmatrix} \frac{3}{5}te^{5t} - \frac{3}{25}e^{5t} - \frac{1}{4}e^{4t} \\ 3te^{2t} - \frac{3}{2}e^{2t} + e^{t}\end{pmatrix} = \begin{pmatrix} \frac{1}{5}te^{5t} - \frac{1}{25}e^{5t} - \frac{1}{12}e^{4t} \\ te^{2t} - \frac{1}{2}e^{2t} + \frac{1}{3}e^{t}\end{pmatrix}$$

$$\varphi\cdot\int\varphi^{-1}\cdot\vec{F} = \begin{pmatrix} e^{-5t} & e^{-2t} \\ -2e^{-5t} & e^{-2t}\end{pmatrix}\begin{pmatrix} \frac{1}{5}te^{5t} - \frac{1}{25}e^{5t} - \frac{1}{12}e^{4t} \\ te^{2t} - \frac{1}{2}e^{2t} + \frac{1}{3}e^{t}\end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5}t - \frac{1}{25} - \frac{1}{12}e^{-t} + t - \frac{1}{2} + \frac{1}{3}e^{-t} \\ -\frac{2}{5}t + \frac{2}{25} + \frac{1}{6}e^{-t} + t - \frac{1}{2} + \frac{1}{3}e^{-t}\end{pmatrix} = \begin{pmatrix} \frac{6}{5}t - \frac{27}{50} + \frac{1}{4}e^{-t} \\ \frac{3}{5}t - \frac{21}{50} + \frac{1}{2}e^{-t}\end{pmatrix}$$

$$X = C_1\begin{pmatrix} 1 \\ -2\end{pmatrix}e^{-5t} + C_2\begin{pmatrix} 1 \\ 1\end{pmatrix}e^{-2t} + \underbrace{\begin{pmatrix} \frac{6}{5}t - \frac{27}{50} + \frac{1}{4}e^{-t} \\ \frac{3}{5}t - \frac{21}{50} + \frac{1}{2}e^{-t}\end{pmatrix}}_{X_p}$$