# Warm-up Problem

Use the Laplace transform to solve the IVP

$$y' - y = 2\cos 5t,\ y(0) = 0$$

# Matrices and Eigenvalues

Ex. Let $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 6 \\ -6 & 10 & -5 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \\ 1 & -1 & 2 \end{pmatrix}$

a) $2A = \begin{pmatrix} 4 & -2 & 6 \\ 0 & 8 & 12 \\ -12 & 20 & -10 \end{pmatrix}$

b) $A + B = \begin{pmatrix} 6 & 6 & -5 \\ 9 & 7 & 11 \\ -5 & 9 & -3 \end{pmatrix}$

<u>Ex.</u> Let $\underset{2\times 2}{A} = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix}$ and $\underset{2\times 2}{B} = \begin{pmatrix} 9 & -2 \\ 6 & 8 \end{pmatrix}$, find $AB$.

$$AB = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 9 & -2 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 36+42 & -8+56 \\ 27+30 & -6+40 \end{pmatrix} = \begin{pmatrix} 78 & 48 \\ 57 & 34 \end{pmatrix}$$

$$\underset{3\times 5}{A} \; \underset{5\times 1}{B} = \underset{3\times 1}{AB}$$

$$\underset{5\times 1}{B} \; \underset{3\times 5}{A} = \, ?$$

$$\langle 3, 1, 7\rangle \cdot \langle -2, 0, 1\rangle$$

$$-6 + 0 + 7$$

$\underline{\text{Ex.}}$ Let $\underset{1\times 3}{A} = \begin{pmatrix} 5 & -6 & 7 \end{pmatrix}$ and $\underset{3\times 1}{B} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$, find $\underset{1\times 1}{AB}$.

$$\underset{1\times 1}{AB} = \begin{pmatrix} 15 - 24 + 7 \end{pmatrix} = \begin{pmatrix} -2 \end{pmatrix}$$

$$\underset{3\times 1}{B}\,\underset{1\times 3}{A} = 3 \times 3$$

**Ex.** $\begin{pmatrix} -4 & 2 \\ 3 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4x+2y \\ 3x+8y \end{pmatrix}$

$2 \times 2 \qquad 2 \times 1 \qquad 2 \times 1$

→ $A^{\mathrm{T}}$ (transpose) switches $a_{ij}$ with $a_{ji}$

Ex. Let $A = \begin{array}{c} \begin{array}{ccc} a_{11} & a_{12} & a_{13} \end{array} \\ \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 6 \\ -6 & 10 & -5 \end{pmatrix} \end{array}$, find $A^{\mathrm{T}}$.

$$A^{T} = \begin{pmatrix} 2 & 0 & -6 \\ -1 & 4 & 10 \\ 3 & 6 & -5 \end{pmatrix}$$

Def. If $AB = I$, then $B$ is the inverse of $A$, written $A^{-1}$.

Thm. If $A$ is an $n \times n$ matrix, $A^{-1}$ exists iff $\det A \neq 0$.

Thm. If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Ex. Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 10 \end{pmatrix}$, find $A^{-1}$.

$$A^{-1} = \frac{1}{10-8}\begin{pmatrix} 10 & -4 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -1 & 1/2 \end{pmatrix}$$

<u>Thm.</u> Let $A$ by $n \times n$ and let $C_{ij} = (-1)^{i+j} M_{ij}$, where $M_{ij}$ is the determinant when row $i$ and column $j$ are removed from $A$. Then

$$A^{-1} = \frac{1}{\det A} C^{\mathrm{T}}$$

Ex. Let $A = \begin{pmatrix} 2 & 2 & 0 \\ -2 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}$, find $A^{-1}$.

$$\left.\begin{array}{lll}
C_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 & C_{12} = (-1)^{1+2}\begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} = 5 & C_{13} = (-1)^{1+3}\begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} = -3 \\
C_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = -2 & C_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = 2 & C_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 2 \\ 3 & 0 \end{vmatrix} = 6 \\
C_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2 & C_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} = -2 & C_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} = 6
\end{array}\right\} C$$

$$\det A = 2\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} = 2(1) - 2(-5) = 12$$

$$A^{-1} = \frac{1}{12}\begin{pmatrix} 1 & -2 & 2 \\ 5 & 2 & -2 \\ -3 & 6 & 6 \end{pmatrix}$$

Ex. Let $X(t)=\begin{pmatrix} \sin 2t \\ e^{3t} \\ 8t-1 \end{pmatrix}$, find $X'(t)$ and $\int X(t)\,dt$

$$X' = \begin{pmatrix} 2\cos 2t \\ 3e^{3t} \\ 8 \end{pmatrix}$$

$$\int X\,dt = \begin{pmatrix} -\frac{1}{2}\cos 2t + A \\ \frac{1}{3}e^{3t} + B \\ 4t^2 - t + C \end{pmatrix}$$

Def. Let $A$ be $n \times n$. A number $\lambda$ is an eigenvalue of $A$ if there is a nonzero vector $K$ such that

$$AK = \lambda K$$

$K$ is called an eigenvector corresponding to $\lambda$.

Ex. Show that $K = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector of

$$A = \begin{pmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{pmatrix}$$

$$AK = \underset{3\times 3}{\begin{pmatrix} 0 & -1 & -3 \\ 2 & 3 & 3 \\ -2 & 1 & 1 \end{pmatrix}} \underset{3\times 1}{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 0+1-3 \\ 2-3+3 \\ -2-1+1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix} = -2K$$

$$\lambda = -2$$

Our equation was $AK = \lambda K$

→ $AK - \lambda K = \vec{0}$

→ $(A - \lambda I)K = \vec{0}$

→ This is only true if $\det(A - \lambda I) = 0$

→ $\det(A - \lambda I) = 0$ is called the <u>characteristic equation</u> of $A$.

<u>Ex.</u> Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} = 1\begin{vmatrix} 6 & -1-\lambda \\ -1 & -2 \end{vmatrix} - 0\begin{vmatrix} \sim & \sim \\ \sim & \sim \end{vmatrix} + (-1-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 6 & -1-\lambda \end{vmatrix}$$

$$= 1\left[-12 - (-1)(-1-\lambda)\right] + (-1-\lambda)\left[(1-\lambda)(-1-\lambda) - 12\right]$$

$$= (-12 - 1 - \lambda) + (-1-\lambda)(\lambda^2 - 1 - 12) = (-13-\lambda) + (-1-\lambda)(\lambda^2 - 13)$$

$$= -13 - \lambda - \lambda^3 - \lambda^2 + 13\lambda + 13 = -\lambda^3 - \lambda^2 + 12\lambda$$

$$= -\lambda(\lambda^2 + \lambda - 12) = -\lambda(\lambda+4)(\lambda-3) = 0$$

$$\lambda_1 = 0 \qquad \lambda_2 = -4 \qquad \lambda_3 = 3$$

$\underline{\lambda_1 = 0:}\ (A - 0I)K = 0$

$$\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \vec{0}$$

$$a + 2b + c = 0$$
$$6a - b = 0$$
$$\to b = 6a$$
$$\to a + 12a + c = 0$$
$$c = -13a$$

$$K_1 = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}$$

$\underline{\lambda_2 = -4}\ (A - (-4)I)K = 0$

$$\begin{pmatrix} 5 & 2 & 1 \\ 6 & 3 & 0 \\ -1 & -2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$6a + 3b = 0 \to b = -2a$$
$$-a - 2b + 3c = 0$$
$$-a + 4a + 3c = 0$$
$$c = -a$$

$$K_2 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

$\underline{\lambda_3 = 3}:\ (A - 3I)K = 0$

$$\begin{pmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ -1 & -2 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$-2a + 2b + c = 0$$
$$6a - 4b = 0 \to a = \tfrac{2}{3}b$$
$$\to -\tfrac{4}{3}b + 2b + c = 0$$
$$c = \tfrac{-2}{3}b$$

$$K_3 = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$$

<u>Ex.</u> Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 3 & 4 \\ -1 & 7 \end{pmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 4 \\ -1 & 7-\lambda \end{vmatrix} = (3-\lambda)(7-\lambda)+4 = \lambda^2-10\lambda+21+4$$

$$= \lambda^2 - 10\lambda + 25 = (\lambda-5)^2 = 0$$

$$\lambda = 5$$

$$(A-5I)\,K = 0$$

$$\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$-a+2b = 0$$

$$a = 2b$$

$$K = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Ex. Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 9 & 1 & 1 \\ 1 & 9 & 1 \\ 1 & 1 & 9 \end{pmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} 9-\lambda & 1 & 1 \\ 1 & 9-\lambda & 1 \\ 1 & 1 & 9-\lambda \end{vmatrix}$$

$$= (9-\lambda)\begin{vmatrix} 9-\lambda & 1 \\ 1 & 9-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 9-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 9-\lambda \\ 1 & 1 \end{vmatrix}$$

$$= (9-\lambda)\left[(9-\lambda)^2-1\right] - \left[9-\lambda-1\right] + \left[1-(9-\lambda)\right]$$

$$= (9-\lambda)(\lambda^2-18\lambda+80) - 8 + \lambda - 8 + \lambda$$

$$= (9-\lambda)(\lambda-8)(\lambda-10) + 2(\lambda-8) = (\lambda-8)\left[(9-\lambda)(\lambda-10)+2\right]$$

$$= (\lambda-8)\left[-\lambda^2+19\lambda-88\right] = -(\lambda-8)(\lambda^2-19\lambda+88)$$

$$= -(\lambda-8)^2(\lambda-11) = 0$$

$$\lambda_1 = 11 \qquad \lambda_2 = 8$$

$\lambda_1 = 11$ : $(A - 11I)K = 0$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\begin{array}{r} -2a + b + c = 0 \\ - \quad a - 2b + c = 0 \\ \hline -3a + 3b \quad = 0 \\ a = b \end{array}$$

$$-2a + a + c = 0$$
$$a = c$$

$$K_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda_2 = 8$ : $(A - 8I)K = 0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$a + b + c = 0$$

$$c = -a - b$$

$$K_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad K_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$