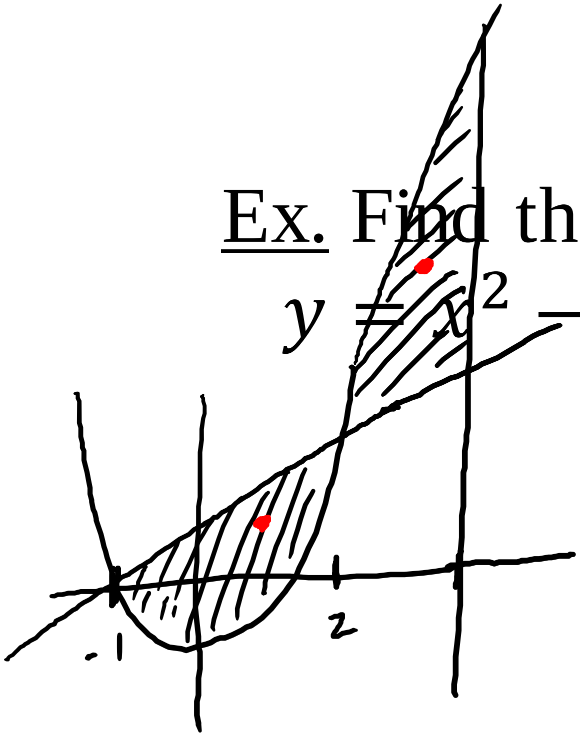
More Areas

$$A=\int_a^b \left(y_{top}-y_{bottom}\right)dx$$

Ex. Find the area of the region bounded by $y=x^2-1$, $y=x+1$, and $x=4$.



$x^2-1=x+1$

$x^2-x-2=0$

$(x-2)(x+1)=0$

$x=2,-1$

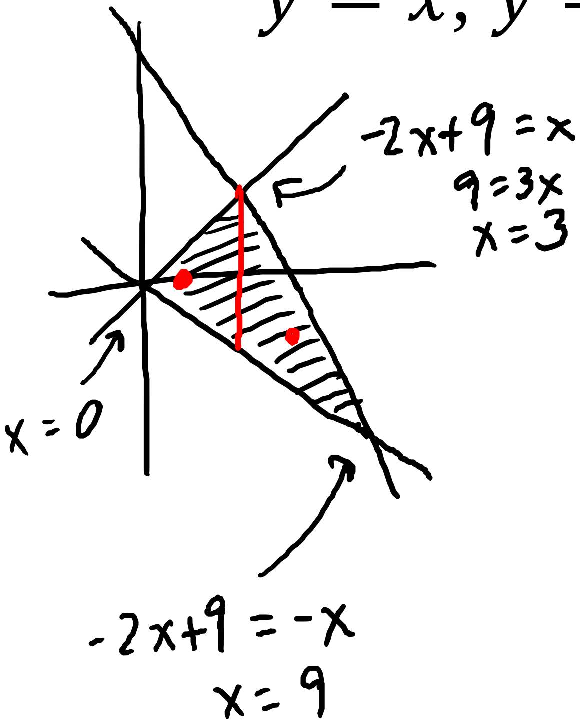
$$A=\int_{-1}^{2}(x+1)-(x^2-1)\,dx+\int_{2}^{4}(x^2-1)-(x+1)\,dx$$

$$=\int_{-1}^{2}(-x^2+x+2)\,dx+\int_{2}^{4}(x^2-x-2)\,dx$$

$$=\left(-\frac{1}{3}x^3+\frac{1}{2}x^2+2x\right)\Big|_{-1}^{2}+\left(\frac{1}{3}x^3-\frac{1}{2}x^2-2x\right)\Big|_{2}^{4}$$

$$=\left(-\frac{8}{3}+2+4\right)-\left(\frac{1}{3}+\frac{1}{2}-2\right)+\left(\frac{64}{3}-8-8\right)-\left(\frac{8}{3}-2-4\right)$$

Ex. Find the area of the region bounded by
$y = x$, $y = -2x + 9$, and $y = -x$.



$$A = \int_0^3 x-(-x)\,dx + \int_3^9 (-2x+9)-(-x)\,dx$$

$$= \int_0^3 2x\,dx + \int_3^9 (-x+9)\,dx$$

$$= x^2\Big|_0^3 + \left(-\frac{x^2}{2} + 9x\right)\Big|_3^9$$

$$= (9-0) + \left(\frac{-81}{2} + 81\right) - \left(\frac{-9}{2} + 27\right)$$

Thm. The average value of a function $f(x)$ over the interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Ex. Find the average value of $f(x) = \sin 5x$ on $[10,30]$.

$$\frac{1}{30-10}\int_{10}^{30} \sin 5x \, dx$$

Ex. The temperature, in $°C$, of a pond is a function W of time t. The table below shows the temperature at selected times. Approximate the average temperature over the time interval $0 \le t \le 15$ using right-hand sums with 5 subintervals.

t	$W(t)$
0	20
3	31
6	28
9	24
12	22
15	21

$$\frac{1}{15-0}\int_0^{15} W(t)\,dt$$

$$= \frac{1}{15}\left[3W(3) + 3W(6) + 3W(9) + 3W(12) + 3W(15)\right]$$

$$= \frac{1}{5}(31+28+24+22+21)\ °C$$

Def. The arc length of a curve on an interval $[a, b]$ is the length of the curve over the interval.

Thm. The arc length, s, of $f(x)$ on $[a, b]$ is given by

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Ex. Find the length of $y = x^{3/2}$ on [0,5].

$$y' = \frac{3}{2}x^{1/2}$$

$$S = \int_a^b \sqrt{1+(f')^2}\,dx = \int_0^5 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx$$

$$= \int_0^5 \sqrt{1+\frac{9}{4}x}\,dx = \int_{\cancel{0}}^{\cancel{5}} u^{1/2}\cdot\frac{4}{9}\,du = \frac{4}{9}\cdot\frac{2}{3}\,u^{3/2}\Big|_{\cancel{*}}^{\cancel{*}}$$

$$\boxed{\begin{aligned} u &= 1+\frac{9}{4}x \\ du &= \frac{9}{4}dx \\ \frac{4}{9}du &= dx \end{aligned}}$$

$$= \frac{8}{27}\left(1+\frac{9}{4}x\right)^{3/2}\Big|_0^5$$

$$= \frac{8}{27}\left(1+\frac{45}{4}\right)^{3/2} - \frac{8}{27}(1)^{3/2}$$

Ex. Find the length of $f(x) = \int_1^x 2\sqrt{4t^2 + 2t}\,dt$ on $[1,2]$.

$$f'(x) = 2\sqrt{4x^2+2x}$$

$$S = \int_1^2 \sqrt{1+\left[2\sqrt{4x^2+2x}\right]^2}\,dx = \int_1^2 \sqrt{1+4(4x^2+2x)}\,dx$$

$$= \int_1^2 \sqrt{16x^2+8x+1}\,dx = \int_1^2 \sqrt{(4x+1)^2}\,dx = \int_1^2 (4x+1)\,dx$$

$$= 2x^2+x\Big|_1^2 = (8+2)-(2+1) = \boxed{7}$$

Ex. Find the length of $y = \frac{x^3}{6} + \frac{1}{2x}$ on [1,2].

$$= \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$$

$$y' = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$$

$$S = \int_1^2 \sqrt{1 + \left(\frac{1}{2}x^2 - \frac{1}{2}x^{-2}\right)^2}\, dx$$