<u>Area</u>

p. 487: 5-6, 9-14, 17-20, 36-39, 49-55 odd, 58, 67, 71-72

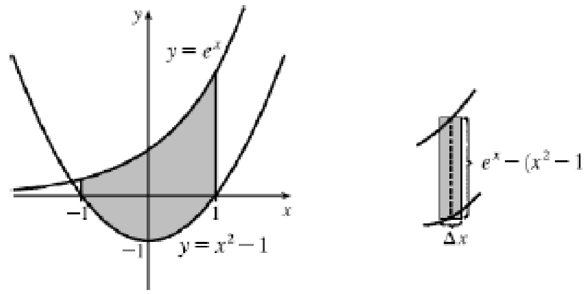
5. $A=\int_{x=1}^{x=8}(y_T-y_B)\,dx=\int_1^8\left(\sqrt[3]{x}-\frac{1}{x}\right)dx=\left[\frac{3}{4}x^{4/3}-\ln|x|\right]_1^8=\left(\frac{3}{4}\cdot 16-\ln 8\right)-\left(\frac{3}{4}-\ln 1\right)=\frac{45}{4}-\ln 8$

6. $A=\int_0^1\left(e^x-xe^{x^2}\right)dx=\left[e^x-\tfrac{1}{2}e^{x^2}\right]_0^1=\left(e-\tfrac{1}{2}e\right)-\left(1-\tfrac{1}{2}\right)=\tfrac{1}{2}e-\tfrac{1}{2}=\tfrac{1}{2}(e-1)$

9. $A=\int_{-1}^{1}(e^x-(x^2-1))\,dx$

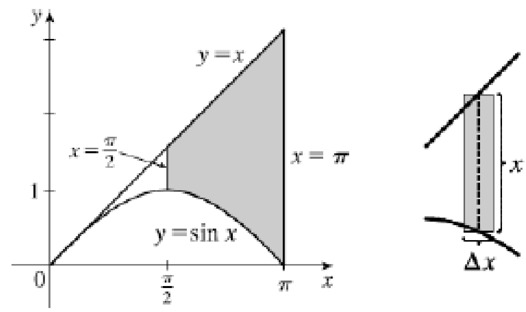
$=\left[e^x-\tfrac{1}{3}x^3+x\right]_{-1}^{1}\quad=(e-\tfrac{1}{3}+1)-(e^{-1}+\tfrac{1}{3}-1)$

$=e-\tfrac{1}{e}+\tfrac{4}{3}$



10. $A=\int_{\pi/2}^{\pi}(x-\sin x)\,dx=\left[\frac{x^2}{2}+\cos x\right]_{\pi/2}^{\pi}$

$=\left(\frac{\pi^2}{2}-1\right)-\left(\frac{\pi^2}{8}+0\right)=\frac{3\pi^2}{8}-1$



11. The curves intersect when $(x-2)^2=x\Leftrightarrow$

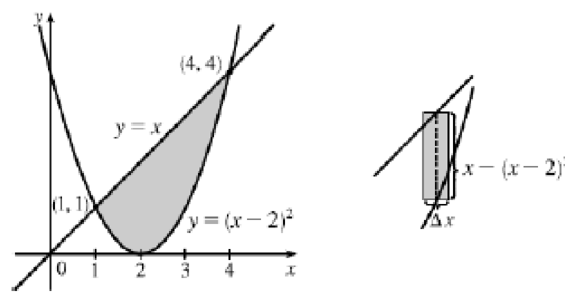
$x^2-4x+4=x\Leftrightarrow x^2-5x+4=0\Leftrightarrow$

$(x-1)(x-4)=0\Leftrightarrow x=1$ or 4.

$A=\int_1^4\left[x-(x-2)^2\right]dx\;=\int_1^4(-x^2+5x-4)\,dx$

$=\left[-\tfrac{1}{3}x^3+\tfrac{5}{2}x^2-4x\right]_1^4$

$=\left(-\tfrac{64}{3}+40-16\right)-\left(-\tfrac{1}{3}+\tfrac{5}{2}-4\right)=\tfrac{9}{2}$



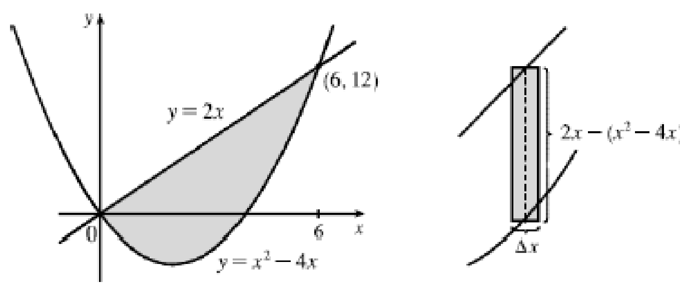
12. The curves intersect when $x^2-4x=2x\Rightarrow$

$x^2-6x=0\Rightarrow x(x-6)=0\Rightarrow x=0$ or 6.

$A=\int_0^6\left[2x-(x^2-4x)\right]dx=\int_0^6(6x-x^2)\,dx$

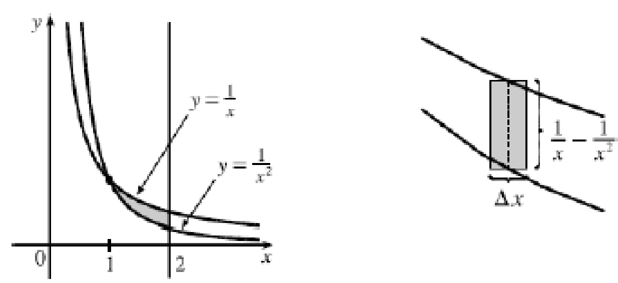
$=\left[3x^2-\tfrac{1}{3}x^3\right]_0^6=\left[3(6)^2-\tfrac{1}{3}(6)^3\right]-(0-0)$

$=108-72=36$



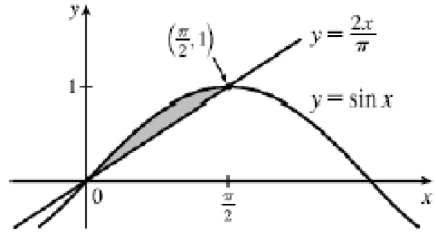
13. $A=\int_1^2\left(\frac{1}{x}-\frac{1}{x^2}\right)dx=\left[\ln x+\frac{1}{x}\right]_1^2$

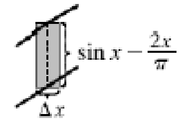
$=\left(\ln 2+\tfrac{1}{2}\right)-(\ln 1+1)=\ln 2-\tfrac{1}{2}$



14. By observation, $y = \sin x$ and $y = 2x/\pi$ intersect at $(0,0)$ and $(\pi/2, 1)$ for $x \geq 0$.

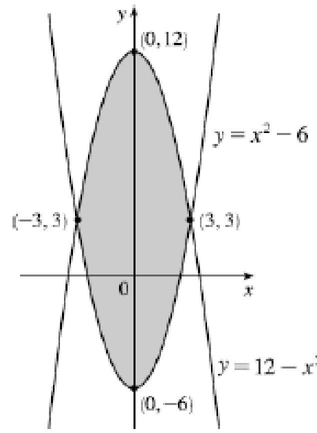
$$A = \int_0^{\pi/2} \left(\sin x - \frac{2x}{\pi} \right) dx = \left[-\cos x - \frac{1}{\pi} x^2 \right]_0^{\pi/2}$$
$$= \left(0 - \frac{\pi}{4} \right) - (-1) = 1 - \frac{\pi}{4}$$





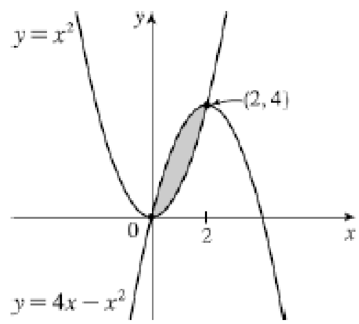
17. $12 - x^2 = x^2 - 6 \Leftrightarrow 2x^2 = 18 \Leftrightarrow$
$x^2 = 9 \Leftrightarrow x = \pm 3$, so

$$A = \int_{-3}^{3} \left[(12 - x^2) - (x^2 - 6) \right] dx = 2\int_0^3 (18 - 2x^2)\, dx \quad \text{[by symmetry]}$$
$$= 2\left[18x - \tfrac{2}{3}x^3 \right]_0^3 = 2[(54 - 18) - 0] = 2(36) = 72$$



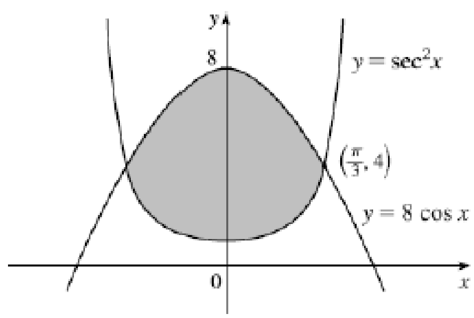
18. $x^2 = 4x - x^2 \Leftrightarrow 2x^2 - 4x = 0 \Leftrightarrow$
$2x(x - 2) = 0 \Leftrightarrow x = 0$ or 2, so

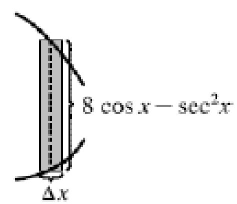
$$A = \int_0^2 \left[(4x - x^2) - x^2 \right] dx = \int_0^2 (4x - 2x^2)\, dx$$
$$= \left[2x^2 - \tfrac{2}{3}x^3 \right]_0^2 = 8 - \tfrac{16}{3} = \tfrac{8}{3}$$



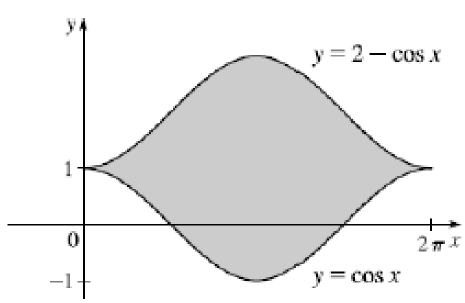
19. The curves intersect when $8\cos x = \sec^2 x \Rightarrow 8\cos^3 x = 1 \Rightarrow \cos^3 x = \frac{1}{8} \Rightarrow \cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}$ for $0 < x < \frac{\pi}{2}$. By symmetry,

$$A = 2\int_0^{\pi/3} (8\cos x - \sec^2 x)\, dx$$
$$= 2\left[8\sin x - \tan x \right]_0^{\pi/3} = 2\left(8 \cdot \tfrac{\sqrt{3}}{2} - \sqrt{3} \right)$$
$$= 2\left(3\sqrt{3} \right) = 6\sqrt{3}$$





20. $$A = \int_0^{2\pi} [(2 - \cos x) - \cos x]\, dx$$
$$= \int_0^{2\pi} (2 - 2\cos x)\, dx$$
$$= [2x - 2\sin x]_0^{2\pi}$$
$$= (4\pi - 0) - 0 = 4\pi$$



36. Setting $f(x) = g(x)$, we find that the curves intersect at $x = -2$ and $x = 3/2$. In the interval $[-2, 3/2]$, $f(x) \geq g(x)$, so the region bounded by the graphs of $f(x)$ and $g(x)$ is
$\int_{-2}^{3/2} (f(x) - g(x))\, dx = \int_{-2}^{3/2} \left(-2x^2 - x + 6 \right) dx = \int_{-2}^{3/2} \left(6 - x - 2x^2 \right) dx$, option (**B**).

37. Let $f(x)=9-x^2$ and $g(x)=x-7$. Then $f(x)=g(x) \Leftrightarrow 9-x^2=x-7 \Leftrightarrow x^2+x-16=0 \Leftrightarrow$

$x=\dfrac{-1\pm\sqrt{1^2+64}}{2}=\dfrac{-1\pm\sqrt{65}}{2}$. Let $a=\dfrac{-1-\sqrt{65}}{2}$ and $b=\dfrac{-1+\sqrt{65}}{2}$. In the interval

(a,b), $f(x)\ge g(x)$, so the area of the region bounded by the graphs of $f(x)$ and $g(x)$ is

$A=\int_a^b(-x^2-x+16)\,dx=\int_b^a(x^2+x-16)\,dx=\left[\frac{1}{3}x^3+\frac{1}{2}x^2-16x\right]_b^a$

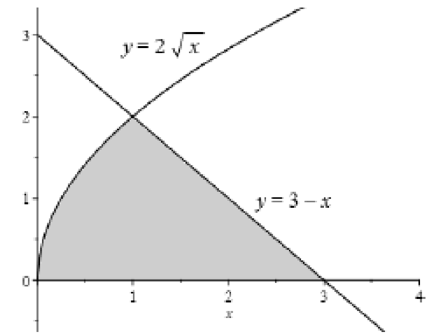
$=\left(\frac{1}{3}a^3+\frac{1}{2}a^2-16a\right)-\left(\frac{1}{3}b^3+\frac{1}{2}b^2-16b\right)=\frac{65}{6}\sqrt{65}$, which is choice (**D**).

38. Let $f(x)=2\sqrt{x}$ and $g(x)=3-x$. Then $f(x)=g(x) \Leftrightarrow 2\sqrt{x}=3-x \Leftrightarrow$
$4x=9-6x+x^2=0 \Leftrightarrow x^2-10x+9=0 \Leftrightarrow (x-1)(x-9)=0$. A graph shows that $g(x)\ge f(x)$ when $0\le x\le 1$, and $f(x)\ge g(x)$ when $1\le x\le 3$. So the area of the region is

$A=\int_0^1\left(3-x-2\sqrt{x}\right)dx+\int_1^3 2\sqrt{x}-(3-x)\,dx$

$=\left[3x-\frac{1}{2}x^2-\frac{4}{3}x^{3/2}\right]_0^1+\left[\frac{4}{3}x^{3/2}-3x+\frac{1}{2}x^2\right]_1^3$

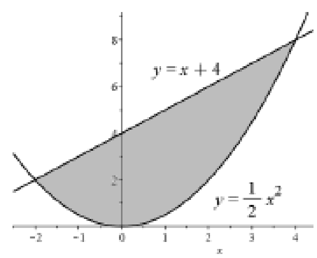
$=\left(1-\frac{1}{2}-\frac{4}{3}\right)-0+\left(\frac{4}{3}\sqrt{9}-9+\frac{9}{2}\right)-\left(\frac{4}{3}-1+\frac{1}{2}\right)=20\sqrt{15}+\frac{9}{2}$



39. The curves intersect at $x=-2$ and $x=4$. The area of the region bounded by the curves is

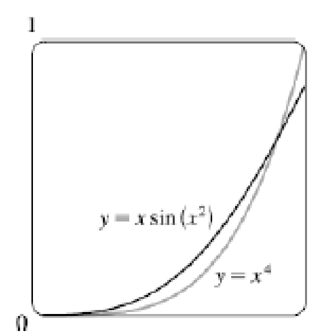
$A=\int_{-2}^4\left(x+4-\frac{1}{2}x^2\right)dx=\left[4x+\frac{1}{2}x^2-\frac{1}{6}x^3\right]_{-2}^4$

$=\left(16+8-\frac{32}{3}\right)-\left(-8+2-\left(-\frac{4}{3}\right)\right)=18$, which is option (**B**).

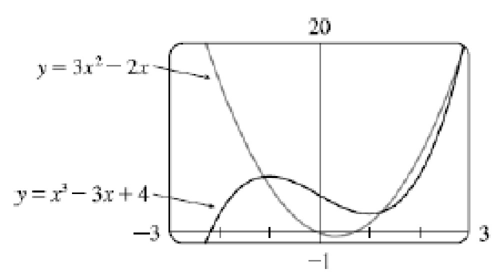


49. From the graph, we see that the curves intersect at $x=0$ and $x=a\approx 0.896$, with $x\sin\left(x^2\right)>x^4$ on $(0,a)$. So the area A of the region bounded by the curves is $A=\int_0^a\left[x\sin(x^2)-x^4\right]dx=\left[-\frac{1}{2}\cos(x^2)-\frac{1}{5}x^5\right]_0^a$

$=-\frac{1}{2}\cos(a^2)-\frac{1}{5}a^5+\frac{1}{2}\approx 0.037$.



51. From the graph, we see that the curves intersect at $x=a\approx -1.114908$, $x=b\approx 1.2541012$, and $x=c\approx 2.860806$, with $x^3-3x+4>3x^2-2x$ on (a,b) and $3x^2-2x>x^3-3x+4$ on (b,c). So the area of the region bounded by the curves is



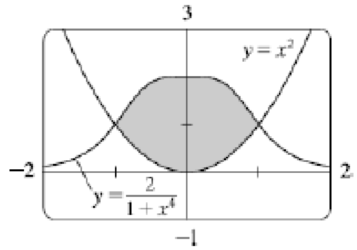
$A=\int_a^b\left[(x^3-3x+4)-(3x^2-2x)\right]dx+\int_b^c\left[(3x^2-2x)-(x^3-3x+4)\right]dx$

$=\int_a^b(x^3-3x^2-x+4)\,dx+\int_b^c(-x^3+3x^2+x-4)\,dx$

$=\left[\frac{1}{4}x^4-x^3-\frac{1}{2}x^2+4x\right]_a^b+\left[-\frac{1}{4}x^4+x^3+\frac{1}{2}x^2-4x\right]_b^c\approx 8.378$.

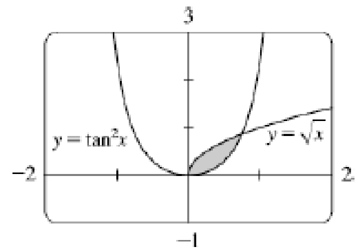
53. Using technology, we can see that the curves intersect at $(-1,1)$ and $(1,1)$. On the interval $(-1,1)$, $\dfrac{2}{1+x^4} > x^2$ so the area of the region is

$$A = \int_{-1}^{1}\left(\frac{2}{1+x^4} - x^2\right)dx \approx 2.80123$$



55. The curves intersect at $x = 0$ and $x = a \approx 0.749363$.

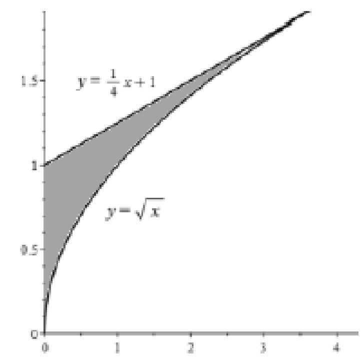
$$A = \int_0^a \left(\sqrt{x} - \tan^2 x\right)dx \approx 0.25142$$



58. (a) $f(x) = \sqrt{x} \Rightarrow f'(x) = \dfrac{1}{2\sqrt{x}} \Rightarrow f'(4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$, so the slope of the tangent line at the point $(4,2)$ is $\frac{1}{4}$ and the equation of the tangent line is $y = \frac{1}{4}(x-4)+2$, or $y = \frac{1}{4}x+1$.

(b) The area of the region bounded by the graph of f, the y-axis and the tangent line is

$$A = \int_0^4 \left(\tfrac{1}{4}x + 1 - \sqrt{x}\right)dx = \left[\tfrac{1}{8}x^2 + x - \tfrac{2}{3}x^{3/2}\right]_0^4 = \left(2 + 4 - \tfrac{2}{3}\cdot 8\right) - 0 = \tfrac{2}{3}$$



67. We know that the area under curve A between $t = 0$ and $t = x$ is $\int_0^x v_A(t)\,dt = s_A(x)$, where $v_A(t)$ is the velocity of car A and s_A is its displacement. Similarly, the area under curve B between $t = 0$ and $t = x$ is $\int_0^x v_B(t)\,dt = s_B(x)$.

(a) After one minute, the area under curve A is greater than the area under curve B. So car A is ahead after one minute.

(b) The area of the shaded region has numerical value $s_A(1) - s_B(1)$, which is the distance by which car A is ahead of car B after 1 minute.

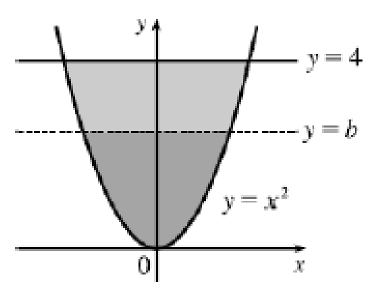
(c) After two minutes, car B is traveling faster than car A and has gained some ground, but the area under curve A from $t = 0$ to $t = 2$ is still greater than the corresponding area for curve B, so car A is still ahead.

(d) From the graph, it appears that the area between curves A and B for $0 \le t \le 1$ (when car A is going faster), which corresponds to the distance by which car A is ahead, seems to be about 3 squares. Therefore, the cars will be side by side at the time x where the area between the curves for $1 \le t \le x$ (when car B is going faster) is the same as the area for $0 \le t \le 1$. From the graph, it appears that this time is $x \approx 2.2$. So the cars are side by side when $t \approx 2.2$ minutes.

71. By the symmetry of the problem, we consider only the first quadrant where $y = x^2 \Rightarrow x = \sqrt{y}$. We are looking for a number b such that

$$\int_0^b \sqrt{y}\,dy = \int_b^4 \sqrt{y}\,dy \Rightarrow \tfrac{2}{3}y^{3/2}\Big]_0^b = \tfrac{2}{3}y^{3/2}\Big]_b^4 \Rightarrow b^{3/2} = 4^{3/2} - b^{3/2}$$

$$\Rightarrow 2b^{3/2} = 8 \Rightarrow b^{3/2} = 4 \Rightarrow b = 4^{3/2} \approx 2.5198.$$

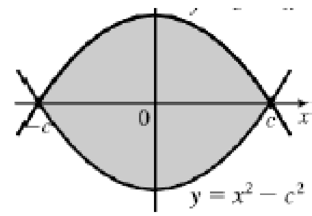


72. (a) We want to choose a so that $\int_1^a \frac{1}{x^2}\,dx = \int_a^4 \frac{1}{x^2}\,dx \Rightarrow$

$\left[\frac{-1}{x}\right]_1^a = \left[\frac{-1}{x}\right]_a^4 \Rightarrow -\frac{1}{a}+1 = -\frac{1}{4}+\frac{1}{a} \Rightarrow \frac{5}{4}=\frac{2}{a} \Rightarrow a = \frac{8}{5}$.

(b) The area under the curve $y = 1/x^2$ from $x=1$ to $x=4$ is $\frac{3}{4}$ [take in the first integral in part (a)]. Now the line $y=b$ must intersect the curve $x = 1/\sqrt{y}$ and not the line $y = 1/4^2$ since the area under the line from $x=1$ to $x=4$ is only $\frac{3}{16}$ which is less than half of $\frac{3}{4}$. We want to choose b so that the upper area in the diagram is half of the total area under the curve $y=1/x^2$ from $x=1$ to $x=4$. This

without changing the graphs, and if $c=0$ the curves do not enclose a region. We see from the graph that the enclosed area A lies between $x=-c$ and $x=c$, and by symmetry, it is equal to four times the area in the first quadrant. The enclosed area is



$A = 4\int_0^c (c^2 - x^2)\,dx = 4\left[c^2x - \frac{1}{3}x^3\right]_0^c = 4\left(c^3 - \frac{1}{3}c^3\right) = 4\left(\frac{2}{3}c^3\right) = \frac{8}{3}c^3$.