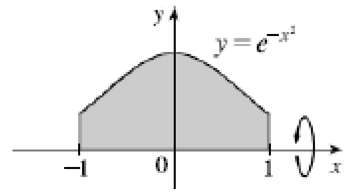
p. 530: 45-46, 49-52, 58-60, 63-64, 74-76

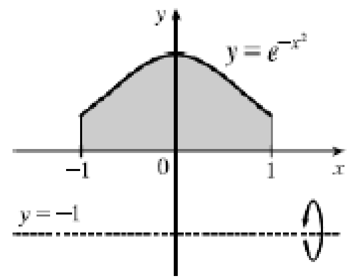
45. (a) About the x-axis:

$$V=\int_{-1}^{1}\pi\left(e^{-x^2}\right)^2dx=2\pi\int_0^1 e^{-2x^2}dx \quad \text{[by symmetry]}$$
$$\approx 3.75825$$



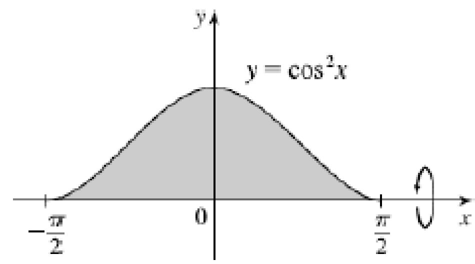
(b) About $y=-1$:

$$V=\int_{-1}^{1}\pi\left\{\left[e^{-x^2}-(-1)\right]^2-\left[0-(-1)\right]^2\right\}dx$$
$$=2\pi\int_0^1\left[\left(e^{-x^2}+1\right)^2-1\right]dx=2\pi\int_0^1\left(e^{-2x^2}+2e^{-x^2}\right)dx$$
$$\approx 13.14312$$



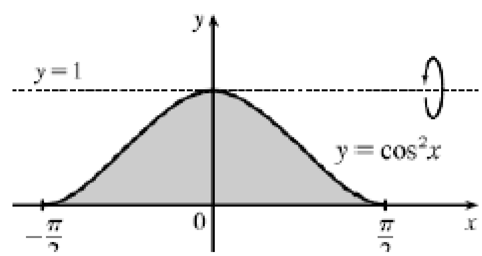
46. (a) About the x-axis:

$$V=\int_{-\pi/2}^{\pi/2}\pi\left(\cos^2 x\right)^2dx=2\pi\int_0^{\pi/2}\cos^4 x\,dx \quad \text{[by symmetry]}$$
$$\approx 3.70110$$



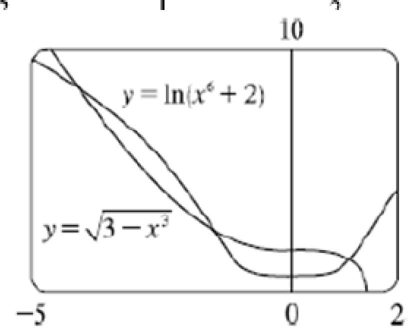
(b) About $y=1$:

$$V=\int_{-\pi/2}^{\pi/2}\pi\left\{[1-0]^2-\left[1-\cos^2 x\right]^2\right\}dx$$
$$=2\pi\int_0^{\pi/2}1-\left(1-2\cos^2 x+\cos^4 x\right)dx$$
$$=2\pi\int_0^{\pi/2}(2\cos^2 x-\cos^4 x)dx\approx 6.16850$$



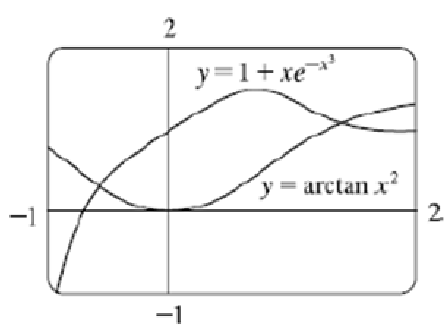
49. $y=\ln(x^6+2)$ and $y=\sqrt{3-x^3}$ intersect at $x=a\approx -4.091$, $x=b\approx -1.467$, and $x=c\approx 1.091$.

$$V=\pi\int_a^b\left\{\left[\ln(x^6+2)\right]^2-\left(\sqrt{3-x^3}\right)^2\right\}dx$$
$$+\pi\int_b^c\left\{\left(\sqrt{3-x^3}\right)^2-\left[\ln(x^6+2)\right]^2\right\}dx\approx 89.023$$

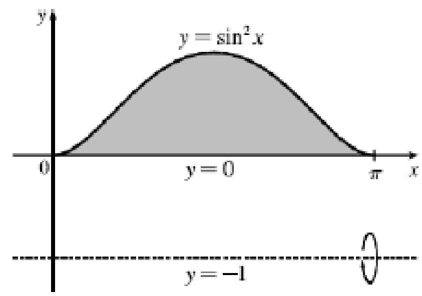


50. $y=1+xe^{-x^3}$ and $y=\arctan x^2$ intersect at $x=a\approx -0.570$, and $x=b\approx 1.391$.

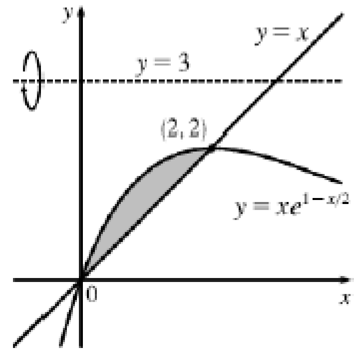
$$V=\pi\int_a^b\left[\left(1+xe^{-x^3}\right)^2-\left(\arctan x^2\right)^2\right]dx\approx 6.923$$



51. $V=\pi\int_0^{\pi}\left\{\left[\sin^2 x-(-1)\right]^2-\left[0-(-1)\right]^2\right\}dx$
$\overset{\text{CAS}}{=}\dfrac{11\pi^2}{8}$

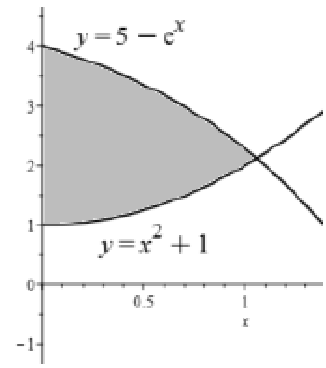


52. $V=\pi\int_0^2\left[(3-x)^2-(3-xe^{1-x/2})^2\right]dx$
$\overset{\text{CAS}}{=}\pi\left(-2e^2+24e-\dfrac{142}{3}\right)$



58. (a) Using technology we find that the curves intersect at $x=a\approx 1.058006401$. The area of the region bounded by these curves is

$$A(x)=\int_0^a 5-e^x-\left(x^2+1\right)dx=\int_0^a\left(4-e^x-x^2\right)dx$$
$$=\left[4x-e^x-\tfrac{1}{3}x^3\right]_0^a=4a-e^a-\tfrac{1}{3}a^3\approx 1.957$$



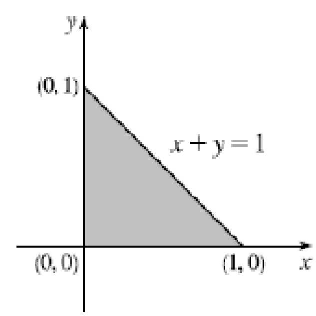
(b) $V=\pi\int_0^a\left(5-e^x\right)^2dx+\pi\int_0^a\left(x^2+1\right)^2dx$

$$=\pi\int_0^a\left(25-10e^x+e^{2x}\right)dx+\pi\int_0^a(x^4+2x^2+1)dx$$
$$=\pi\left[25x-10e^x+\tfrac{1}{2}e^{2x}\right]_0^a+\pi\left[\tfrac{1}{5}x^5+\tfrac{2}{3}x^3+x\right]_0^a$$
$$=\pi\left(25a-10e^a+\tfrac{1}{2}e^a\right)-(-10+\tfrac{1}{2})+\pi\left(\tfrac{1}{5}a^5+\tfrac{2}{3}a^3+a\right)-0\approx 9.180245\pi\approx 28.841.$$

(c) The cross-section of the base corresponding to the coordinate x has length $y=4-e^x-x^2$. The corresponding semicircle with diameter $2y$ has area

$$A(x)=\frac{1}{3}\pi\left(\frac{4-e^x-x^2}{2}\right)^2=\frac{1}{12}\pi\left(4-e^x-x^2\right)^2$$ Therefore,

$$V=\int_0^a\frac{1}{12}\pi\left(4-e^x-x^2\right)^2dx\overset{\text{CAS}}{\approx}0.3683\pi\approx 1.1571.$$



(d) $\int_0^k(5-e^x)-(x^2+1)\,dx=\int_k^a(5-e^x)-(x^2+1)\,dx$

59. (a) The area of R is $\int_0^{\pi/2} 6\sin 2x\,dx = -3\cos 2x\Big]_0^{\pi/2} = -3(-1-1) = 6$, and the resulting solid has volume

$$V = \pi\int_0^{\pi/2}(6\sin 2x)^2\,dx \overset{\text{CAS}}{=} = 9\pi^2$$

(b) Each cross-section is an equilateral triangle with side length $s = 6\sin 2x$, and area

$A(x) = \dfrac{\sqrt{3}}{4}(6\sin 2x)^2 = 9\sqrt{3}\sin^2 2x$. Therefore,

$$V = \int_0^{\pi/2} 9\sqrt{3}\sin^2 2x\,dx = 9\sqrt{3}\int_0^{\pi/2}\sin^2 2x\,dx \overset{\text{CAS}}{=} 9\sqrt{3}\left(\frac{\pi}{4}\right) = \frac{9\sqrt{3}}{4}\pi$$

(c) A cross-section is a washer with inner radius $0-(-3) = 3$ and outer radius $6\sin 2x - (-3)$ $= 6\sin 2x + 3$. The volume of this solid is $V = \pi\int_0^{\pi/2}\left[(6\sin 2x - 3)^2 - 3^2\right]dx \overset{\text{CAS}}{=} 36\pi + \frac{27}{2}\pi^2$.

60. $V = \int_0^{10} A(x)\,dx \approx M_5 = \frac{10-0}{5}\left[A(1)+A(3)+A(5)+A(7)+A(9)\right]$

$= 2(0.65+0.61+0.59+0.55+0.50) = 2(2.90) = 5.8 \text{ m}^3$

63. (a) The area of the region R is the area of the rectangle of height 4 and length 2, minus the area under the parabola from $x = 0$ to $x = 2$, that is $A = 2\cdot 4 - \int_0^2 x^2\,dx = 8 - \frac{1}{3}x^3\Big]_0^2 = 8 - \left(\frac{8}{3} - 0\right) = \frac{16}{3}$.

(b) The cross-section of the base corresponding to the coordinate y has length $x = \sqrt{y}$. The corresponding semicircle with radius $\frac{1}{2}x = \frac{1}{2}\sqrt{y}$ has area $A(y) = \frac{1}{2}\cdot\pi\left(\frac{1}{2}\sqrt{y}\right)^2 = \frac{\pi}{2}\cdot\frac{1}{4}y = \frac{\pi}{8}y$.

Therefore $V = \int_0^2 \frac{\pi}{8}y\,dy = \frac{\pi}{8}\cdot\frac{1}{2}y^2\Big]_0^2 = \frac{\pi}{16}\cdot(4-0) = \frac{\pi}{4}$.

(c) A cross-section is a washer with an inner radius of $7-4=3$ and outer radius $7-x^2$. Therefore the volume of this solid is

$$V = \pi\int_0^2\left[(7-x^2)^2 - 3^2\right]dx = \pi\int_0^2\left(x^4 - 14x^2 + 40\right)dx = \pi\left[\tfrac{1}{5}x^5 - \tfrac{14}{3}x^3 + 40x\right]_0^2 = \tfrac{736}{15}\pi$$

64. If the sides of the rectangle R are parallel to the coordinate axes, and a diagonal had endpoints $(2,3)$ and $(7,5)$, then the rectangle has corners at $(2,3), (7,3), (2,5)$, and $(7,5)$. The side parallel to the x-axis therefore has length 5 and the side parallel to the y-axis has length 2. The equilateral triangle that has a side on in R, perpendicular to the y-axis, therefore has side length $l = 5$. The area of the triangle is therefore $\frac{\sqrt{3}}{4}5^2 = \frac{25\sqrt{3}}{4}$. The volume of the resulting solid of revolution is

$V = \int_3^5 \frac{25\sqrt{3}}{4}\,dy = \left[\frac{25\sqrt{3}}{4}y\right]_3^5 = \frac{25\sqrt{3}}{4}(5-3) = \frac{25\sqrt{3}}{2}$. It is not possible to calculate the volume of the solid by using cross-sections perpendicular to the x-axis.

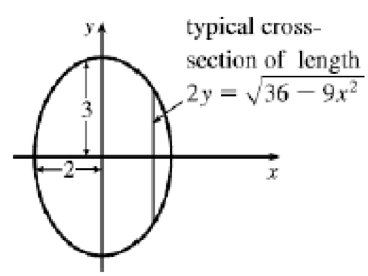
74. If l is a leg of the isosceles right triangle and $2y$ is the hypotenuse, then

$l^2 + l^2 = (2y)^2 \Rightarrow 2l^2 = 4y^2 \Rightarrow l^2 = 2y^2$.

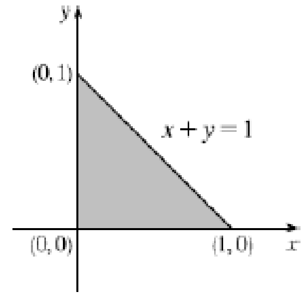
$V = \int_{-2}^2 A(x)\,dx = 2\int_0^2 A(x)\,dx = 2\int_0^2 \frac{1}{2}l\cdot l\,dx = 2\int_0^2 y^2\,dx$

$= 2\int_0^2 \frac{1}{4}\left(36 - 9x^2\right)dx = \frac{9}{2}\int_0^2\left(4 - x^2\right)dx$

$= \frac{9}{2}\left[4x - \frac{1}{3}x^3\right]_0^2 = \frac{9}{2}\left(8 - \frac{8}{3}\right) = 24$



75. (a) The cross-section of the base corresponding to the coordinate y has length $x = 1 - y$. The corresponding equilateral triangle with side s has area $A(y) = s^2\left(\frac{\sqrt{3}}{4}\right) = (1-y)^2\left(\frac{\sqrt{3}}{4}\right)$. Therefore



$$V = \int_0^1 A(y)\,dy = \int_0^1 (1-y)^2\left(\frac{\sqrt{3}}{4}\right)dy$$
$$= \frac{\sqrt{3}}{4}\int_0^1\left(1-2y+y^2\right)dy = \frac{\sqrt{3}}{4}\left[y - y^2 + \tfrac{1}{3}y^3\right]_0^1$$
$$= \frac{\sqrt{3}}{4}\left(\frac{1}{3} = \right)\frac{\sqrt{3}}{12}$$

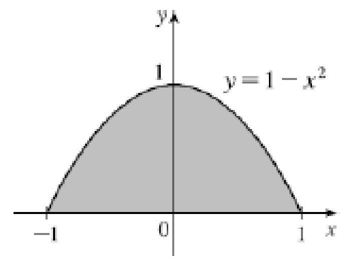
Or: $\displaystyle\int_0^1 (1-y)^2\left(\frac{\sqrt{3}}{4}\right)dy = \frac{\sqrt{3}}{4}\int_1^0 u^2(-du)\ [u = 1-y] = \frac{\sqrt{3}}{4}\left[\frac{1}{3}u^3\right]_0^1 = \frac{\sqrt{3}}{12}$

(b) The cross-section of the base corresponding to the coordinate x has length $y = 1 - x$. The corresponding square with side s has area $A(x) = s^2 = \frac{1}{2}(1-x)^2 = 1 - 2x + x^2$. Therefore,

$$V = \int_0^1 A(x)\,dx = \int_0^1\left(1-2x+x^2\right)dx = \left[x - x^2 + \tfrac{1}{3}x^3\right]_0^1 = \left(1 - 1 + \tfrac{1}{3}\right) - 0 = \tfrac{1}{3}$$

Or: $\int_0^1 (1-x)^2\,dx = \int_1^0 u^2(-du)\ \ [u = 1-x] = \frac{1}{3}u^3\Big]_0^1 = \frac{1}{3}$

76. (a) The cross-section of the base corresponding to the coordinate y has length $2x = 2\sqrt{1-y}$. $\left[y = 1 - x^2 \Leftrightarrow x = \pm\sqrt{1-y}\right]$ The corresponding square with side s has area $A(x) = s^2 = \left(2\sqrt{1-y}\right)^2 = 4(1-y)$. Therefore,



$$V = \int_0^1 A(y)\,dy = \int_0^1 4(1-y)\,dy = 4\left[y - \tfrac{1}{2}y^2\right]_0^1 = 4\left[\left(1 - \tfrac{1}{2}\right) - 0\right] = 2.$$

(b) The cross-section of the base b corresponding to the coordinate x has length $1 - x^2$. The height h also has length $1 - x^2$, so the corresponding isosceles triangle has area $A(x) = \frac{1}{2}bh = \frac{1}{2}\left(1-x^2\right)^2$. Therefore,

$$V = \int_{-1}^1 A(x)\,dx = \int_{-1}^1 \tfrac{1}{2}(1-x^2)^2\,dx = 2\cdot\tfrac{1}{2}\int_0^1 (1 - 2x^2 + x^4)\,dx \quad \text{[by symmetry]}$$
$$= \left[x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\right]_0^1 = \left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right) - 0 = \tfrac{8}{15}$$