<u>DEs and Slope Fields</u>

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5. $\dfrac{dy}{dx}=3x^2y^2 \Rightarrow \dfrac{dy}{y^2}=3x^2\,dx \Rightarrow \displaystyle\int\frac{dy}{y^2}=\int 3x^2\,dx\ \ [y\neq 0] \Rightarrow \int y^{-2}\,dy=\int 3x^2\,dx \Rightarrow -y^{-1}=x^3+C \Rightarrow$
$\dfrac{-1}{y}=x^3+C \Rightarrow y=\dfrac{-1}{x^3+C}$. $y=0$ is also a solution.

7. $xyy'=x^2+1 \Rightarrow xy\dfrac{dy}{dx}=x^2+1 \Rightarrow y\,dy=\dfrac{x^2+1}{x}dx\ \ [x\neq 0] \Rightarrow \displaystyle\int y\,dy=\int\left(x+\frac{1}{x}\right)dx \Rightarrow$
$\frac{1}{2}y^2=\frac{1}{2}x^2+\ln|x|+K \Rightarrow y^2=x^2+2\ln|x|+2k \Rightarrow y=\pm\sqrt{x^2+2\ln|x|+C}$, where $C=2K$.

9. $(e^y-1)y'=2+\cos x \Rightarrow (e^y-1)\dfrac{dy}{dx}=2+\cos x \Rightarrow (e^y-1)dy=(2+\cos x)\,dx \Rightarrow$
$\int\left(e^y-1\right)dy=\int(2+\cos x)\,dx \Rightarrow e^y-y=2x+\sin x+C$. We cannot solve explicitly for y.

14. $\dfrac{dz}{dt}+e^{t+z}=0 \Rightarrow \dfrac{dz}{dt}=-e^te^z \Rightarrow \int e^{-z}\,dz=\int e^t\,dt \Rightarrow -e^{-z}=-e^t+C \Rightarrow e^{-z}=e^t-C \Rightarrow$
$\dfrac{1}{e^z}=e^t-C \Rightarrow e^z=\dfrac{1}{e^t-C} \Rightarrow z=\ln\left(\dfrac{1}{e^t-C}\right) \Rightarrow z=-\ln\left(e^t-C\right)$

15. $\dfrac{dy}{dx}=ky \Rightarrow \displaystyle\int\frac{dy}{y}=\int k\,dx \Rightarrow \ln|y|=kx+C_1 \Rightarrow y=e^{kx+C_1}=Ce^{kx}$, equation (**C**).

16. $\int dy=\int 2x\,dx \Rightarrow y=f(x)=x^2+C$. $f(1)=3 \Rightarrow 3=(1)^2+C \Rightarrow C=1$. Therefore, $f(x)=x^2+2$, option (**B**).

17. $y=e^{x/2} \Rightarrow \dfrac{dy}{dx}=\frac{1}{2}e^{x/2} \Rightarrow 2\,dy=e^{x/2}\,dx \Rightarrow 2\dfrac{dy}{e^{x/2}}=dx \Rightarrow 2\dfrac{dy}{y}=dx$, which is choice, (**A**).

18. Observe that the graphs of equations (B), (C) and (D) go through (–1, 3), so they all satisfy the initial condition. $\dfrac{dy}{dx}=(y-1)(2x+1) \Rightarrow \displaystyle\int\frac{dy}{y-1}=\int(2x+1)dx \Rightarrow \ln|y-1|=x^2+x+K \Rightarrow$
$|y-1|=Ce^{x^2+x} \Rightarrow y=Ce^{x^2+x}+1$. The initial condition $y(-1)=3$ forces $C=2$, so equation (**D**) is the desired solution.

19. $f'(x)=x\sqrt{f(x)} \Rightarrow \dfrac{dy}{dx}=xy^{1/2} \Rightarrow \int y^{-1/2}\,dy=\int x\,dx \Rightarrow 2y^{1/2}=\frac{1}{2}x^2+K \Rightarrow \sqrt{y}=\frac{1}{4}x^2+C.$
$f(3)=25 \Rightarrow \sqrt{25}=\left(\frac{1}{4}\cdot 3^2+C\right) \Rightarrow \frac{11}{4}=C$. Therefore, $y=\left(\frac{1}{4}x^2+\frac{11}{4}\right)^2$, so $y(0)=\left(\frac{11}{4}\right)^2=\frac{121}{16}$, which is option (**D**).

20. $\dfrac{dy}{dx}=xe^y \Rightarrow \int e^{-y}dy=\int x\,dx \Rightarrow -e^{-y}=\frac{1}{2}x^2+C.$ $y(0)=0 \Rightarrow -e^0=\frac{1}{2}(0)^2+C \Rightarrow$
$C=-1$, so $-e^{-y}=\frac{1}{2}x^2-1 \Rightarrow e^{-y}=-\frac{1}{2}x^2+1 \Rightarrow y=\ln\left(1-\frac{1}{2}x^2\right) \Rightarrow y=-\ln\left(1-\frac{1}{2}x^2\right).$

22. $\dfrac{du}{dt}=\dfrac{2t+\sec^2 t}{2u}$, $u(0)=-5$. $\int 2u\,du=\int(2t+\sec^2 t)\,dt \Rightarrow u^2=t^2+\tan t+C$, where
$[u(0)]^2=0^2+\tan 0+C \Rightarrow C=(-5)^2=25$. Therefore, $u^2=t^2+\tan t+25$, so $u=\pm\sqrt{t^2+\tan t+25}$.
Because $u(0)=-5<0$, we must have, $u=-\sqrt{t^2+\tan t+25}$.

28. $\frac{dy}{dx}=\frac{x}{y} \Rightarrow y\,dy = x\,dx \Rightarrow \int y\,dy = \int x\,dx \Rightarrow \frac{1}{2}y^2=\frac{1}{2}x^2+C \Rightarrow\ y(0)=2 \Rightarrow \frac{1}{2}(2)^2=\frac{1}{2}(0)^2+C \Rightarrow$

$C=2$, so $\frac{1}{2}y^2=\frac{1}{2}x^2+2 \Rightarrow y^2=x^2+4 \Rightarrow y=\sqrt{x^2+4}$ since $y(0)=2>0$.

29. $f'(x)=xf(x)-x \Rightarrow \frac{dy}{dx}=xy-x \Rightarrow \frac{dy}{dx}=x(y-1) \Rightarrow \frac{dy}{y-1}=x\,dx \quad [y\neq 1] \Rightarrow$

$\int \frac{dy}{y-1}=\int x\,dx \Rightarrow \ln|y-1|=\frac{1}{2}x^2+C.$ $f(0)=2 \Rightarrow \ln|2-1|=\frac{1}{2}(0)^2+C \Rightarrow C=0$, so

$\ln|y-1|=\frac{1}{2}x^2 \Rightarrow |y-1|=e^{x^2/2} \Rightarrow y-1=e^{x^2/2}$ $[\text{since } f(0)=2] \Rightarrow y=e^{x^2/2}+1$

42. (a) For this differential equation, when $x=0 \Rightarrow \frac{dy}{dx}-0=0 \Rightarrow \frac{dy}{dx}=0,$ so the slope of the tangent line is zero. Therefore, when $x=0$, the tangent line must be horizontal.

(b) $\frac{dy}{dx}-2xy=x \Rightarrow \frac{dy}{dx}=x(2y+1) \Rightarrow \int\frac{dy}{2y+1}=\int x\,dx \Rightarrow \frac{1}{2}\ln|2y+1|=\frac{1}{2}x^2+C \Rightarrow$

$\ln|2y+1|=x^2+C \Rightarrow 2y+1=e^{x^2+C} \Rightarrow y+1=\frac{1}{2}Ae^{x^2} \Rightarrow y=\frac{1}{2}\left(Ae^{x^2}-1\right)$

(c) $y(0)=0 \Rightarrow 0=\frac{1}{2}\left(Ae^0-1\right) \Rightarrow A=1,$ so the solution is $y=\frac{1}{2}(e^{x^2}-1)$

43. $f'(x)=2x[f(x)]^2 \Rightarrow \frac{dy}{dx}=2xy^2 \Rightarrow \int y^{-2}dy=\int 2x\,dx \Rightarrow -y^{-1}=x^2+C \Rightarrow y=-\left(\frac{1}{x^2+C}\right).$

$f(0)=2=-\left(\frac{1}{0^2+C}\right) \Rightarrow C=-\frac{1}{2} \Rightarrow y=\frac{-2}{2x^2-1}.$ Therefore, $f(2)=\frac{-2}{2(2)^2-1}=-\frac{2}{7}$, **(B)**.

44. (a) At this point, the tangent line has slope $\left.\frac{dy}{dx}\right|_{(x,y)=(0,1)}=4-1=-3.$ Therefore, the equation of the tangent line at (0, 1) is $y-1=3(x-0)$, or $y=3x+1$. Using this tangent line we can approximate $f(1.5)\approx 3(1.5)+1=5.5.$

(b) $\frac{dy}{dx}=4-y \Rightarrow \int\frac{dy}{4-y}=\int dx \Rightarrow \ln|4-y|=x+C \Rightarrow |4-y|=Ae^x \Rightarrow 4-y=Ae^x \Rightarrow$

$f(x)=4-Ae^x$. Then $\lim_{x\to 0}\frac{f(x)}{6x}=\lim_{x\to 0}\frac{4-Ae^x}{6x}=\frac{4-A}{\lim_{x\to 0} 6}=\infty$, so the limit does not exist.

(c) From (b), the general solution is $y=4\pm Ae^x$. $f(0)=1 \Rightarrow 1=4-A\cdot e^0 \Rightarrow A=3$. Thus, $y=f(x)=4-3e^{-x}$.

48. (a) When $t = 1$, the tangent line has slope $\frac{dW}{dt} = \frac{1}{5}(80-30) = \frac{50}{5} = 10 \Rightarrow$ the equation of the tangent line at (0, 30) is $y = 10t + 30$. We can use the tangent line to approximate $W(1) \approx 10 + 30 = 40$ g.

(b) $\frac{d^2W}{dt^2} = -\frac{1}{5}\cdot\frac{dW}{dt} = -\frac{1}{5}\left[\frac{1}{5}(80-W)\right] = -\frac{1}{25}(80-W)$; Because the second derivative is negative at the point (0, 30), the graph of W is concave down, so the tangent line lies above the curve and the tangent line estimate is an overestimate.

(c) $\frac{dW}{dt} = \frac{1}{5}(80-W) \Rightarrow \int \frac{dW}{(80-W)} = \int \frac{1}{5}dt \Rightarrow \ln|80-W| = \frac{1}{5}t + C \Rightarrow |80-W| = Ae^{0.2t} \Rightarrow$

$80 - W = Ae^{0.2t} \Rightarrow W(t) = 80 - Ae^{0.2t}$.

$W(0) = 30 \Rightarrow 30 = 80 + Ae^{0} \Rightarrow A = -50 \Rightarrow W(t) = 80 - 50e^{0.2t}$.

49. (a) $\frac{dy}{dx} = \frac{2}{xy} = 2(xy)^{-1} \Rightarrow \frac{d^2y}{dx^2} = -2(xy)^{-2}\left(x\frac{dy}{dx} + y\right) = \frac{-2}{x^2y^2}\left(x\frac{2}{xy} + y\right) = \frac{-2}{x^2y^2}\left(\frac{2+y^2}{y}\right) = \frac{-4-2y^2}{x^2y^3}$.

(b) At the point (1, 3), the tangent line has slope $\frac{2}{1\cdot 3} = \frac{2}{3}$. The equation of the tangent line at this point is $y = \frac{2}{3}(x-1) + 3 = \frac{2}{3}x + \frac{7}{3}$. We can approximate $f(1.2) \approx \frac{2}{3}(1.2) + \frac{7}{3} = \frac{47}{15} \approx$ 3.133

(c) If $f(x) > 0$ for $1 \le x \le 1.5$, by (a), $f''(x) = \frac{-4-2y^2}{x^2y^3} < 0$ on this interval, which means that the curve is concave down on this interval, and the tangent line lies above the curve, so the tangent line estimate is an overestimate.

(d) $\frac{dy}{dx} = \frac{2}{xy} \Rightarrow \int y\,dy = \int 2x^{-1}\,dx \Rightarrow \frac{1}{2}y^2 = 2\ln|x| + C \Rightarrow y^2 = 4\ln|x| + c$.

$y(1) = 3 \Rightarrow 3^2 = 4\ln|1| + c \Rightarrow 9 = c \Rightarrow y^2 = 4\ln x + 9 \Rightarrow y = \sqrt{4\ln x + 9}$.

50. $x^2 f'(x) = f(x) \Rightarrow x^2\frac{dy}{dx} = y \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x^2} \Rightarrow \ln|y| = -\frac{1}{x} + C \Rightarrow y = ce^{-1/x}$, and $f(1) = \frac{2}{e} \Rightarrow$ $\frac{2}{e} = \frac{c}{e} \Rightarrow c = 2,\ y = 2e^{-1/x}$. For this function, $f(-1) = 2e$, $f(2) = \frac{2}{\sqrt{e}}$, $\lim_{x\to 0} f(x)$ does not exist because the limit from the left does not equal the limit from the right. However, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{2}{e^{1/x}} = 2$, so only (**B**) is true.

51. (a) $\frac{dy}{dt} = k(y-70) \Rightarrow \int \frac{dy}{y-70} = \int k\,dt \Rightarrow \ln|y-70| = kt + C_1 \Rightarrow y - 70 = Ce^{kt} \Rightarrow y = Ce^{kt} + 70$.

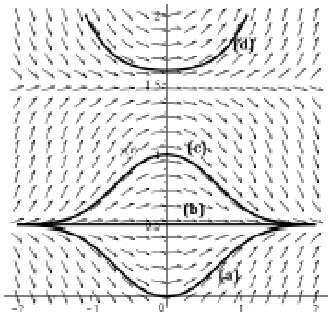
$y(0) = 88 \Rightarrow 88 = C + 70 \Rightarrow C = 18 \Rightarrow y = 18e^{kt}$.

$y(2) = 85 \Rightarrow 85 = 18e^{2k} + 70 \Rightarrow 15/18 = e^{2k} \Rightarrow \ln(15/18) = 2k \Rightarrow k = \frac{1}{2}\ln\frac{5}{6}$

(b) $y = 70 + 18e^{(1/2)\ln(5/6)t}$

(c) $75 = 70 + 18e^{(1/2)\ln(5/6)t} \Rightarrow \frac{5}{18} = e^{0.5\ln(5/6)t} \Rightarrow 2\cdot\ln\left(\frac{5}{18}\right) = \ln\left(\frac{5}{6}\right)t \Rightarrow t = \frac{2\cdot\ln\left(\frac{5}{18}\right)}{\ln\left(\frac{5}{6}\right)} \approx 14.051$ min.

p. 615: 7-11, 13-16, 19-23 odd, 40-41, 43 (a-b)

7. 

It appears that the constant functions $y = 0.5$ and $y = 1.5$ are equilibrium solutions. Note that these two values of y satisfy the given differential equation $y' = x\cos \pi y$.

8. The function F depends on both x and y because the line segments along each horizontal are not parallel (do not have the same slope), nor do the line segments along each vertical line. The line segments in the fourth quadrant have negative slopes, so (C) is not true. However, the horizontal line segments occur for some x in $[-1, 1]$, which indicates that $F = 0$ for some x in this interval. Thus, **D** is the true statement.

9. For $y > 0$, the given slope field indicates that F does not depend on y, and has horizontal tangents at $x = 1$. In addition, the slopes are positive for $x > 1$ and negative for $x < 1$, with the absolute value of the slopes decreasing as x approaches 1. The differential equation corresponding to the general solution $y = \pm(x-1)^2 + C$ would be $y = \pm 2(x-1)^2$. This means that option (**A**), $y = \pm(x-1)^2 + C$ could be the general solution for this slope field.
This slope field for $y > 0$ appears to be reflected across the y-axis to obtain the slope field for $y < 0$. Thus, for $y < 0$, the solution could be $y = -(x-1)^2 + C$.

10. The slope field indicates that the differential equation must involve both x and y, so possibilities (A) and (B) are eliminated. At the point (1, 4), the slope field has a line segment with a negative slope. This is true for option (**C**), but not for option (D).

11. In the given slope field, we see that the solution curve passing through (0, 1) has slopes that are positive and increasing, so (D) is true. For $y = 2e^x - x - 1 \Rightarrow \frac{dy}{dx} = 2e^2 - 1 = x + \left(2e^x - x - 1\right) = x + y$, so (C) is true. The solution passing through (0, 1) does appear to be asymptotic to the graph of $y = x - 1$, so (B) is true. However, if $y = x - 1$, then $x + y = 2x + 1 \neq 1 = \frac{dy}{dx}$, so statement **A** is not true.

13. $y' = 2 - y$. The slopes at each point are independent of x, so the slopes are the same along each line parallel to the x-axis. Thus, **III** is the direction field for this equation. Note that for $y = 2$, $y' = 0$.

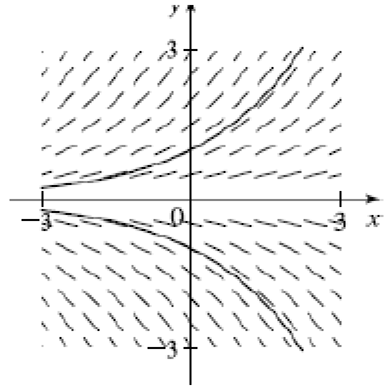
14. $y' = x(2-y) = 0$ on the lines $x = 0$ and $y = 2$. Direction field **I** satisfies these conditions.

15. $y' = x + y - x = 0$ on the line $y = -x + 1$. Direction field **IV** satisfies these conditions. Observe also that on the line $y = -x$ we have $y' = -1$, which is true in IV.

16. $y' = \sin x \sin y = 0$ on the lines $x = 0$, $y = 0$ and $y' > 0$ 0 for $0 < x < \pi$, $0 < y < \pi$. Direction field **II** satisfies these conditions.

19.

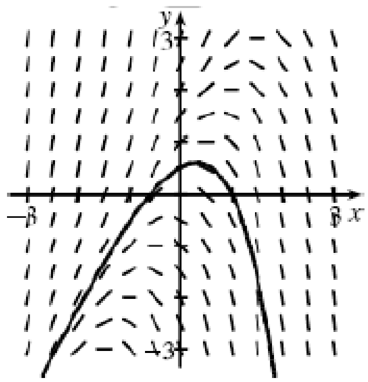
x	y	$y' = \frac{1}{2}y$
0	0	0
0	1	0.5
0	2	1
0	–3	–1.5
0	–2	–1



Note that the three solution curves sketched go through $(0,0), (0,1)$, and $(0,-1)$.

21.

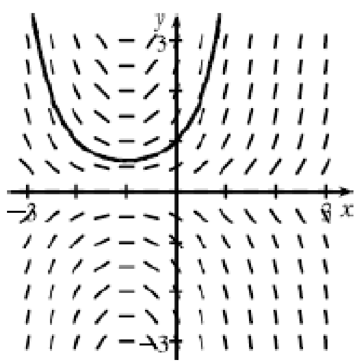
x	y	$y' = y - 2x$
–2	–2	2
–2	2	6
2	2	–2
2	–2	–6



Note that $y' = 0$ for any point on the line $y = 2x$. The slopes are positive to the left of the line and negative to the right of the line. The solution curve in the graph passes through (1, 0).

23.

x	y	$y' = y + xy$
0	±1	±2
1	±2	±4
–3	±2	±4



Note that $y' = y(x+1) = 0$ for any point on $y = 0$ or on $x = -1$. The slopes are positive when the factors y and have the same sign and negative when they have opposite signs. The solution curve in the graph passes through $(0,1)$.

40. (a) $y = 6x + b \Rightarrow \frac{dy}{dx} = 6.$ We need y to be a solution to the differential equation, so

$$6 = \frac{dy}{dx} = 2y - 12x = 2(6x + b) - 12x \Rightarrow 6 = 12x + 2b - 12x = 2b \Rightarrow b = 3.$$

(b) $\frac{dy}{dx} = g' \Rightarrow g'(0,0) = 2(0) - 12(0) = 0,$ thus, g has a critical point at $(0,0)$.

$\frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 12 = 2(2y - 12x) - 12 = 4y - 24x - 12 \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{(x,y)=(0,0)} = -12.$ Therefore g is concave down at this point and the graph of g must have a relative maximum at $(0,0)$.

41. $f(x) = y = -2x + 4e^{-x} + c \Rightarrow f'(x) = \frac{dy}{dx} = -2 - 4e^{-x} \Rightarrow$

$\frac{dy}{dx} + y = -2 - 4e^{-x} + (-2x + 4e^{-x} + c) = c - 2 - 2x = -2x + (c - 2).$ In order for f to be a solution of the differential equation we must have $c - 2 = 0 \Rightarrow c = 2,$ which is option (**C**).

43. (a) If f has a critical point at $x = \ln 2,$ f must have a horizontal (or undefined) tangent line at this point. Therefore, $\left.\frac{dy}{dx}\right|_{(x,y)=(\ln 2,y)} = 2x - y\big|_{(x,y)=(\ln 2,y)} = 0 \Rightarrow 2(\ln 2) - y = 0 \Rightarrow y = 2\ln 2.$

(b) $\frac{dy}{dx} = 2x - y \Rightarrow \frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y) = 2 - 2x + y.$ At the point $(\ln 2, 2\ln 2),$ $2 - 2(\ln 2) + 2\ln y = 2 > 0,$ so this point is a relative minimum.