Higher Order Derivatives

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41. If $f(x)=2x^3-3x^2$ then using the limit definition we find

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{2(x+h)^3-3(x+h)^2-\left(2x^3-3x^2\right)}{h}$$

$$=\lim_{h\to 0}\frac{2x^3+6x^2h+6xh^2+2h^3-3x^2-6xh-3h^2-2x^3+3x^2}{h}$$

$$=\lim_{h\to 0}\frac{6x^2h+6xh^2+2h^3-6xh-3h^2}{h}=\lim_{h\to 0}\frac{h\left(6x^2+6xh+2h^2-6x-3h\right)}{h}$$

$$=\lim_{h\to 0}\left(4x^2+4xh+2h^2-6x-3h\right)=6x^2-6x.$$

Then using the limit definition again,

$$f''(x)=\lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}=\lim_{h\to 0}\frac{6(x+h)^2-6(x+h)-\left(6x^2-6x\right)}{h}$$

$$=\lim_{h\to 0}\frac{6x^2+12xh+6h^2-6x-6h-6x^2+6x}{h}=\lim_{h\to 0}\frac{12xh+6h^2-6h}{h}=\lim_{h\to 0}\frac{h(12x+6h-6)}{h}$$

$$=\lim_{h\to 0}(12x+6h-6)=12x-6$$ which is (**D**).

42. If $f(x)=x^3$, then $f'(x)=3x^2$, and $f''(x)=6x$, and $f'''(x)=6$, which is a constant function so $f^{(iv)}(x)=0$ which is option (**D**).

59. Call the curve with the positive y-intercept g and the other curve h. Notice that g has a maximum (horizontal tangent) at $x=0$, but $h\neq 0$, so h cannot be the derivative of g. Also notice that where g is positive, h is increasing. Thus $h=f$ and $g=f'$. Now $f'(-1)$ is negative since f' is below the x-axis there and $f''(1)$ is positive since f is concave upward at $x=1$. Therefore, $f''(1)$ is greater than $f'(-1)$.

60. Call the curve with the smallest positive x-intercept g and the other curve h. Notice that where g is positive in the first quadrant, h is increasing. Thus $h=f$ and $g=f'$. Now $f'(-1)$ is positive since f' is above the x-axis there and $f''(1)$ appears to be zero since f has an inflection point at $x=1$. Therefore, $f'(1)$ is greater than $f''(-1)$.

61. $a=f, b=f', c=f''$. We can see this because where a has a horizontal tangent, $b=0$, and where b has a horizontal tangent, $c=0$. We can immediately see that c can be neither f nor f' since at the points where c has a horizontal tangent, neither a nor b is equal to 0.

63. We can see immediately that a is the graph of the acceleration function, since at the points where a has a horizontal tangent, neither c nor b is equal to 0. Next, we note that $a=0$ at the point where b has a horizontal tangent, so b must be the graph of the velocity function, and hence $b'=a$. We conclude that c is the graph of the position function.

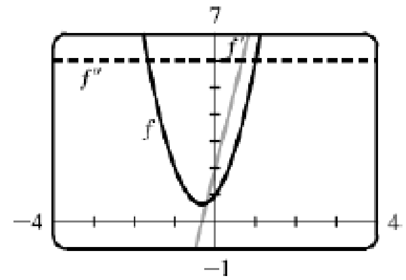
65. $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{3(x+h)^2+2(x+h)+1-\left(3x^2+2x+1\right)}{h}$

$= \lim_{h\to 0}\frac{3x^2+6xh+3h^2+2x+2h+1-3x^2-2x-1}{h} = \lim_{h\to 0}\frac{6xh+3h^2+2h}{h}$

$= \lim_{h\to 0}\frac{h\left(6x+3h+2\right)}{h} = \lim_{h\to 0}\left(6x+3h+2\right) = 6x+2$

$f''(x) = \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h} = \lim_{h\to 0}\frac{6(x+h)+2-(6x+2)}{h}$

$= \lim_{h\to 0}\frac{6x+6h+2-6x-2)}{h} = \lim_{h\to 0}\frac{6h}{h} = \lim_{h\to 0} 6 = 6$



We see from the graph that our answers are reasonable because the graph of f' is that of a linear function and the graph of f'' is that of a constant function.

66. $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^3-3(x+h)-\left(x^3-3x\right)}{h}$

$= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-3x-3h-x^3+3x}{h} = \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-3h}{h}$

$= \lim_{h\to 0}\frac{h\left(3x^2+3xh+h^2-3\right)}{h} = \lim_{h\to 0}\left(3x^2+3xh+h^2-3\right) = 3x^2-3$

$f''(x) = \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h} = \lim_{h\to 0}\frac{3(x+h)^2-3-(3x^2-3)}{h} = \lim_{h\to 0}\frac{3x^2+6xh+3h^2-3-3x^2+3}{h}$

$= \lim_{h\to 0}\frac{6xh+3h^2}{h} = \lim_{h\to 0}\frac{h\left(6x+3h\right)}{h} = \lim_{h\to 0}\left(6x+3h\right) = 6x$

We see from the graph that our answers are reasonable because the graph of f' is that of an even function (f is an odd function) and the graph of f'' is that of an odd function. Furthermore, $f' = 0$ when f has a horizontal tangent and $f'' = 0$ when f' has a horizontal tangent.