Implicit Differentiation

p. 248: 9-11, 19-35 odd, 36-37, 55-61 odd, 63-70, 79-80

9. $y^2+x=5 \Rightarrow 2yy'+1=0 \Rightarrow y'=-\frac{1}{2y} \Rightarrow$ the slope of the tangent line where $y=2$ is $=-\frac{1}{2(2)}=-\frac{1}{4}$, which is option (**C**).

10. $x^2+y^2=a^2 \Rightarrow 2x+2yy'=0 \Rightarrow y'=-\frac{2x}{2y}=-\frac{x}{y}$, which is option (**B**).

11. $x^2-y^2=25 \Rightarrow 2x-2yy'=0 \Rightarrow y'=\frac{x}{y} \Rightarrow$

$$y''=\frac{y(1)-xy'}{y^2}=\frac{y-x\frac{x}{y}}{y^2}=\frac{y-\frac{x^2}{y}}{y^2}=\frac{\frac{y^2-(25+y^2)}{y}}{y^2}=-\frac{25}{y^3}$$ which is option (**B**).

19. $\frac{d}{dx}(2x^2+xy-y^2)=\frac{d}{dx}(2) \Rightarrow 4x+xy'+y(1)-2yy'=0 \Rightarrow xy'-2yy'=-4x-y \Rightarrow$

$(x-2y)y'=-4x-y \Rightarrow y'=\frac{-4x-y}{x-2y}$

21. $\frac{d}{dx}(x^3-xy^2+y^3)=\frac{d}{dx}(1) \Rightarrow 3x^2-x\cdot 2yy'-y^2(1)+3y^2y'=0 \Rightarrow 3y^2y'-2xyy'=y^2-3x^2 \Rightarrow$

$(3y^2-2xy)y'=y^2-3x^2 \Rightarrow y'=\frac{y^2-3x^2}{3y^2-2xy}=-\frac{y^2-3x^2}{y(3y-2x)}$

23. $\frac{d}{dx}(xe^y)=\frac{d}{dx}(x-y) \Rightarrow xe^yy'-e^y\cdot 1=1-e^x \Rightarrow xe^yy'+y'=1-e^y \Rightarrow y'(xe^y+1)=1-e^y \Rightarrow$

$y'=\frac{1-e^y}{xe^y+1}$

25. $\frac{d}{dx}(\cos xy)=\frac{d}{dx}(1+\sin y) \Rightarrow (-\sin xy)(xy'+y)=\cos y\cdot y' \Rightarrow -xy'\sin(xy)-\cos y\cdot y'=y\sin(xy)$

$\Rightarrow y'[-x\sin(xy)-\cos y]=y\sin(xy) \Rightarrow y'=\frac{y\sin(xy)}{-x\sin(xy)-\cos y}=-\frac{y\sin(xy)}{x\sin(xy)+\cos y}$

27. $\frac{d}{dx}(e^y\sin x)=\frac{d}{dx}(x+xy) \Rightarrow e^y\cos x+\sin xe^yy'=1+xy'+y\cdot 1 \Rightarrow$

$e^y\sin x\cdot y'-xy'=1+y-e^y\cos x \Rightarrow y'(e^y\sin x-x)=1+y-e^y\cos x \Rightarrow y'=\frac{1+y-e^y\cos x}{e^y\sin x-x}$

29. $\frac{d}{dx}(xy)=\frac{d}{dx}\sqrt{x^2+y^2} \Rightarrow xy'+y(1)=\frac{1}{2}(x^2+y^2)^{-1/2}(2x+2yy') \Rightarrow$

$xy'+y=\frac{x}{\sqrt{x^2+y^2}}+\frac{y}{\sqrt{x^2+y^2}}y' \Rightarrow xy'-\frac{y}{\sqrt{x^2+y^2}}y'=\frac{x}{\sqrt{x^2+y^2}}-y \Rightarrow$

$\frac{x\sqrt{x^2+y^2}-y}{\sqrt{x^2+y^2}}y'=\frac{x-y\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} \Rightarrow y'=\frac{x-y\sqrt{x^2+y^2}}{x\sqrt{x^2+y^2}-y}$

31. $\frac{d}{dx}(x\sin y + y\sin x) = \frac{d}{dx}(1) \Rightarrow x\cos y \cdot y' + \sin y \cdot 1 + y\cos x + \sin x \cdot y' = 0 \Rightarrow$
$x\cos y \cdot y' + \sin x \cdot y' = -\sin y - y\cos x \Rightarrow y'(x\cos y + \sin x) = -\sin y - y\cos x \Rightarrow$
$y' = \frac{-\sin y - y\cos x}{x\cos y + \sin x}$

33. $\tan(x-y) = \frac{y}{1+x^2} \Rightarrow (1+x^2)\tan(x-y) = y \Rightarrow (1+x^2)\sec^2(x-y)\cdot(1-y') + \tan(x-y)\cdot 2x = y' \Rightarrow$
$(1+x^2)\sec^2(x-y) - (1+x^2)\sec^2(x-y)y' + 2x\tan(x-y) = y' \Rightarrow$
$(1+x^2)\sec^2(x-y) + 2x\tan(x-y) = [1+(1+x^2)\sec^2(x-y)]\cdot y' \Rightarrow$
$y' = \frac{(1+x^2)\sec^2(x-y) + 2x\tan(x-y)}{1+(1+x^2)\sec^2(x-y)}$

35. $2xe^x + 2ye^y = 4 \Rightarrow 2x\cdot e^x + 2\cdot e^x + (2y\cdot e^y\cdot y' + e^y\cdot 2\cdot y') = 0 \Rightarrow$
$ye^y y' + e^y y' = -(xe^x + e^x) \Rightarrow (ye^y + e^y)y' = -(xe^x + e^x) \Rightarrow y' = \frac{-(xe^x + e^x)}{ye^y + e^y}$

36. $\frac{d}{dx}\{f(x) + x^2[f(x)]^3\} = \frac{d}{dx}(10) \Rightarrow f'(x) + x^2\cdot 3[f(x)]^2\cdot f'(x) + [f(x)]^3\cdot 2x = 0.$ If $x=1$, we have
$f'(1) + 1^2\cdot 3[f(1)]^2\cdot f'(1) + [f(1)]^3\cdot 2(1) = 0 \Rightarrow f'(1) + 1^2\cdot 3\cdot 2^2\cdot f'(1) + 2^3\cdot 2 = 0 \Rightarrow$
$f'(1) + 12f'(1) = -16 \Rightarrow 13f'(1) = -16 \Rightarrow f'(1) = -\frac{16}{13}.$

37. $\frac{d}{dx}[g(x) + x\sin g(x)] = \frac{d}{dx}(x^2) \Rightarrow g'(x) + x\cos g(x)\cdot g'(x) + \sin g(x)\cdot 1 = 2x.$ If $x=0$, we have
$g'(0) + 0 + \sin g(0) = 2(0) \Rightarrow g'(0) + \sin 0 = 0 \Rightarrow g'(0) + 0 = 0 \Rightarrow g'(0) = 0.$

55. $y\sin 2x = x\cos 2y \Rightarrow y\cdot\cos 2x\cdot 2 + \sin 2x\cdot y' = x(-2\sin 2y\cdot 2y') + \cos(2y)\cdot 1 \Rightarrow$
$\sin 2x\cdot y' + 2x\sin 2y\cdot y' = -2y\cos 2x + \cos 2y \Rightarrow y'(\sin 2x + 2x\sin 2y) = -2y\cos 2x + \cos 2y \Rightarrow$
$y' = \frac{-2y\cos 2x + \cos 2y}{\sin 2x + 2x\sin 2y}$. When $x = \frac{\pi}{2}$ and $y = \frac{\pi}{4}$, we have $y' = \frac{-(\pi/2)(-1) + 0}{0 + \pi\cdot 1} = \frac{\pi/2}{\pi} = \frac{1}{2}$, so an equation of the tangent line is $y = \frac{1}{2}(x - \frac{\pi}{2}) + \frac{\pi}{4}$ or $y = \frac{1}{2}x$.

57. $x^2 - xy - y^2 = 1 \Rightarrow 2x - (xy' + y\cdot 1) - 2yy' = 0 \Rightarrow 2x - xy' - y - 2yy' = 0 \Rightarrow 2x - y = xy' + 2yy' \Rightarrow$
$2x - y = (x+2y)y' \Rightarrow y' = \frac{2x-y}{x+2y}$. When $x=2$ and $y=1$, we have $y' = \frac{4-1}{2+2} = \frac{3}{4}$, so an equation of the tangent line is $y = \frac{3}{4}(x-2) + 1$ or $y = \frac{3}{4}x - \frac{1}{2}$.

59. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \Rightarrow 2x + 2yy' = 2(2x^2 + 2y^2 - x)(4x + 4yy' - 1)$. When $x = 0$ and $y = \frac{1}{2}$, we have $0 + y' = 2\left(\frac{1}{2}\right)(2y' - 1) \Rightarrow y' = 2y' - 1 \Rightarrow y' = 1$, so an equation of the tangent line is $y = 1(x-0) + \frac{1}{2}$ or $y = x + \frac{1}{2}$.

61. $2\left(x^2+y^2\right)^2=25(x^2-y^2)\Rightarrow 4\left(x^2+y^2\right)(2x+2yy')=25(2x-2yy')\Rightarrow$

$4(x+yy')(x^2+y^2)=25(x-yy')\Rightarrow 4yy'(x^2+y^2)+25yy'=25x-4x(x^2+y^2)\Rightarrow$

$y'=\dfrac{25x-4x(x^2+y^2)}{25y+4y(x^2+y^2)}$. When $x=3$ and $y=1$, we have $y'=\frac{75-120}{24+40}=-\frac{45}{65}=-\frac{9}{13}$, so an equation of the tangent line is $y=-\frac{9}{13}(x-3)+1$ or $y=-\frac{9}{13}x+\frac{40}{13}$.

63. $\tan(xy)=x\Rightarrow \sec^2(xy)[xy'+y(1)]=1\Rightarrow x\sec^2(xy)y'=1-y\sec^2(xy)\Rightarrow$

$$y'=\frac{1-y\sec^2(xy)}{x\sec^2(xy)}=\frac{1-\frac{y}{\cos^2(xy)}}{\frac{x}{\cos^2(xy)}}\cdot\left[\frac{\cos^2(xy)}{\cos^2(xy)}\right]=\frac{\cos^2(xy)-y}{x},$$ which is option (**A**).

64. $x^2-3xy+y^2=-20\Rightarrow 2x-3xy'-3y+2yy'=0\Rightarrow y'(3x+2y)=3y-2x\Rightarrow$

$y'=\dfrac{3y-2x}{3x+2y}$. At the point $(6,4)$, $y'=\dfrac{3(4)-2(6)}{3(6)+2(4)}=0$ so the tangent line is horizontal and the equation of the tangent line is (**A**) $y=4$.

65. $x^2-12x^3y+y^3=-15\Rightarrow 2x-12x^3y'-36x^2y+3yy'=0\Rightarrow\ 2x-36x^2y=\left(12x^3-3y^2\right)y'\Rightarrow$

$\dfrac{2x-36x^2y}{12x^3-3y^2}=y'$. At the point $(1,2)$, the slope of the tangent line is undefined, and the tangent line is vertical. Thus the equation of the tangent line is (**A**), $x=1$.

66. $3x+xy+y^2=9\Rightarrow 3+(xy'+y)+2yy'=0\Rightarrow xy'+2yy'=-3-y\Rightarrow$

$(x+2y)y'=-3-y\Rightarrow y'=\dfrac{-3-y}{x+2y}$. When $y=2$, $3x+xy+y^2=9\Rightarrow 3x+x(2)+2^2=9\Rightarrow$

$5x=9-4\Rightarrow x=1$. And when $y=2$, the slope of the tangent line is $y'=\dfrac{-3-2}{1+2(2)}=-\dfrac{5}{5}=-1$. Then the slope of the normal line is $-\left(\dfrac{1}{-1}\right)=1$, option (**C**).

67. $x^2+xy+y^2=7\Rightarrow\ 2x+(xy'+y)+2yy'=0\Rightarrow xy'+2yy'=-2x-y\Rightarrow$

$(x+2y)y'=-2x-y\Rightarrow y'=\dfrac{-2x-y}{x+2y}$. The slope at the point $(2,1)$ is $\dfrac{-2(2)-1}{2+2(1)}=-\dfrac{5}{4}$ and this must equal y'. So $-\dfrac{5}{4}=\dfrac{-2x-y}{x+2y}\Rightarrow 5x+10y=8x+4y\Rightarrow 6y=3x\Rightarrow 2y=x$. Substituting this for x in $x^2+xy+y^2=7$ we have $4y^2+2y^2+y^2=7\Rightarrow 7y^2=7\Rightarrow y=\pm1$. Then $x=2y=2(-1)=-2$, so the point on the curve with the same slope is $(-2,-1)$.

68. $e^y=\sin x\Rightarrow e^y\dfrac{dy}{dx}=\cos x=\Rightarrow\ \dfrac{dy}{dx}=\dfrac{\cos x}{e^y}=\dfrac{\cos x}{\sin x}=\tan x$, option (**B**).

69. $\dfrac{dy}{dx}=\sqrt{25-y^2}=(25-y^2)^{1/2}\Rightarrow\dfrac{d^2y}{dx^2}=\dfrac{1}{2}(25-y^2)^{-1/2}\cdot(-2y)\left(\dfrac{dy}{dx}\right)=-\dfrac{y}{\sqrt{25-y^2}}\cdot\sqrt{25-y^2}=-y.$

When $y=3$, $\dfrac{d^2y}{dx^2}=-3$, which is choice (**B**).

70. $\tan^{-1}\left(\frac{2}{x}\right)=\tan^{-1}(2x^{-1})=\frac{N}{2}\Rightarrow\frac{1}{1+(2x^{-1})^2}\left(-\frac{1}{x^2}\right)\frac{dx}{dN}=\frac{1}{2}\Rightarrow\frac{dx}{dN}=-\frac{1}{2}x^2\left(1+\frac{4}{x^2}\right)=-\frac{1}{2}x^2-2.$

When $x=2$, $\frac{dx}{dN}=-\frac{1}{2}(2)^2-2=-2-2=-4$, option (**B**).

79. If $x=0$ in $xy+e^y=e$, then we get $0+e^y=e$, so $y=1$ and the point where $x=0$ is $(0,1)$. Differentiating implicitly with respect to x gives us $xy'+y\cdot 1+e^y y'=0$. Substituting 0 for x and 1 for y gives us $0+1+ey'=0\Rightarrow ey'=-1\Rightarrow y'=-1/e$. Differentiating $xy'+y\cdot 1+e^y y'=0$ implicitly with respect to x gives us $xy''+y'\cdot 1+y'+e^y y''+y'\cdot e^y y'=0$. Now substitute 0 for x, 1 for y and $-1/e$ for y'. $0+\left(-\frac{1}{e}\right)+\left(-\frac{1}{e}\right)+ey''+\left(-\frac{1}{e}\right)(e)\left(-\frac{1}{e}\right)=0\Rightarrow-\frac{2}{e}+ey''+\frac{1}{e}=0\Rightarrow ey''=\frac{1}{e}\Rightarrow y''=\frac{1}{e^2}.$

80. If $x=1$ in $x^2+xy+y^3=1$, then we get $1+y+y^3=1\Rightarrow y^3+y=0\Rightarrow y(y^2+1)=0\Rightarrow y=0$, so the point where $x=1$ is $(1,0)$. Differentiating $x^2+xy+y^3=1$, implicitly with respect to x gives us $2x+xy'+y\cdot 1+3y^2y'=0$. Substituting 1 for x and 0 for y gives us $2+y'+0+0=0\Rightarrow y'=-2$. Differentiating $2x+xy'+y\cdot 1+3y^2y'=0$ implicitly with respect to x gives us $2+xy''+y'\cdot 1+y'+3(y^2y''+y'\cdot 2yy')=0$. Now substitute 1 for x, and 0 for y, and -2 for y' to get $2+y''+(-2)+(-2)+3(0+0)=0\Rightarrow y''=2$.