Optimization

p. 374: 6-8, 11-14, 20, 24, 48

6. Call the two numbers $x+100$ and x. Minimize $f(x)=(x+100)x=x^2+100x$.
$f'(x)=2x+100=0 \Rightarrow x=-50$. Since $f''(x)=2>0$, there is an absolute minimum at $x=-50$. The two numbers are 50 and -50.

7. Call the two numbers x and $\dfrac{100}{x}$, where $x>0$. Minimize $f(x)=x+\dfrac{100}{x}$. $f'(x)=1-\dfrac{100}{x^2}=\dfrac{x^2-100}{x^2}$.
The critical number is $x=10$. Since $f'(x)<0$ for $0<x<10$ and $f'(x)>0$ for $x>10$, there is an absolute minimum at $x=10$. The numbers are 10 and 10.

8. Call the two numbers x and y. Then $x+y=16$, so $y=16-x$. Call the sum of their squares S. Then
$S=x^2+y^2=x^2+(16-x)^2 \Rightarrow S'=2x+2(16-x)(-1)=2x-32+2x=4x-32$. $S'=0 \Rightarrow x=8$.
Since $S'(x)<0$ for $0<x<8$ and $S'(x)>0$ for $x>8$, there is an absolute minimum at $x=8$. Thus,
$y=1608=8$ and $S^2=8^2+8^2=128$.

11. If the rectangle has dimensions x and y, then its perimeter is $2x+2y=100$ m, so $y=50-x$. Thus, the area is $A=xy=x(50-x)$. We wish to maximize the function $A=xy=x(50-x)=50x-x^2$, where $0<x<50$. Since $A'(x)=50-2x=-2(x-25)$, $A'(x)>0$ for $0<x<25$ and $A'(x)<0$ for $25<x<50$. Thus, A has an absolute maximum at $x=25$, and $A(25)=25^2=625$ m^2. The dimensions of the rectangle that maximize its area are $x=y=25$ m. (The rectangle is a square.)

12. If the rectangle has dimensions x and y, then its area is $xy=1000\text{m}^2$, so $y=1000/x$. The perimeter is $P=2x+2y=2x+2000/x$. Thus, the area is $A=xy=x(50-x)$. We wish to minimize the function $P(x)=2x+2000/x$ for $x>0$. Since $P'(x)=2-2000/x^2=(2/x^2)(x^2-1000)$, so the only critical number in the domain of P is $x=\sqrt{1000}$. $P''(x)=4000/x^3>0$, so P is concave up throughout its domain and $P\left(\sqrt{1000}\right)=4\sqrt{1000}$ is an absolute minimum value. The dimensions of the rectangle with minimal perimeter are $x=y=\sqrt{1000}=10\sqrt{10}$ m. (The rectangle is a square.)

13. If a rectangle has its base along the x-axis and two vertices on the parabola $y=16-x^2$, then one side of the parabola has length $2x$, and the other has height $y=16-x^2$, so the area of the rectangle is $2x(16-x^2)$. This is option (**C**).

14. The square of the distance d between the point $(3,-1)$ and a point $(x,2x+1)$ on the line $y=2x+1$ is
$S=d^2=(x-3)^2+(y-(-1))^2==(x-3)^2+(2x+1+1)^2=5x^2+2x+13$
Then $S'(x)=10x+2=0 \Leftrightarrow 10x=-2 \Leftrightarrow x=-\frac{1}{5}$. And, $S''(x)=10>0$, so the lone critical point is a minimum. Therefore, the closest point is $\left(-\frac{1}{5},2\left(-\frac{1}{5}\right)+1\right)=\left(-\frac{1}{5},\frac{3}{5}\right)$ which is choice (**A**).

20. (a) Let x be the length of a side of the square base of the box and h be the height of the box. Then The area of the box is $x \cdot x + 4 \cdot xh = 108 \text{ m}^2 \Rightarrow 4xh = 108 - x^2 \Rightarrow x \Rightarrow h = \frac{108}{4x} - \frac{1}{4}x \Rightarrow h = \frac{27}{x} - \frac{1}{4}x.$

Then the volume of the box is $V(x) = x^2 h = x^2\left(\frac{27}{x} - \frac{1}{4}x\right) = 27x - \frac{1}{4}x^3.$

(b) We need to maximize the volume, so $V'(x) = 27 - \frac{3}{4}x^2 = 0 \Leftrightarrow 27 = \frac{3}{4}x^2 \Leftrightarrow 36 = x^2 \Leftrightarrow 6 = x$ (since $x > 0$). Then $V''(x) = -\frac{3}{2}x \Rightarrow V''(6) = -9 < 0 \Rightarrow$ the volume of the box is maximized when $x = 6$ m. So the dimensions that maximize the volume of the box are $x = 6$ m by 6 m by $h = \frac{27}{6} - \frac{6}{4} = 3$ m.

24. Let b be the length of the base of the box and h the height. The surface area is $1200 = b^2 + 4bh$ $\Rightarrow h = (1200 - b^2)/(4b)$. The volume is $V = b^2 h = b^2(1200 - b^2)/(4b) = 300b - b^3/4 \Rightarrow$ $V'(b) = 300 - \frac{3}{4}b^2$. $V'(b) = 0 \Rightarrow 300 = \frac{3}{4}b^2 \Rightarrow b^2 = 400 \Rightarrow b = \sqrt{400} = 20$. Since $V'(b) > 0$ for $0 < b < 20$ and $V'(b) < 0$ for $b > 20$, there is an absolute maximum when $b = 20$ by the First Derivative Test for Absolute Extreme Values. If $b = 20$, then $h = (1200 - 20^2)(4 \cdot 20) = 10$, so the largest possible volume is $b^2 h = (20)^2(10) = 4000 \text{ cm}^3$.

48. $xy = 384 \Leftrightarrow y = 384/x$. The total area is $A(x) = (8 + x)(12 + 384/x) = 12(40 + x + 256/x)$, so $A'(x) = 12(1 - 256/x^2) = 0 \Rightarrow x = 16$. There is an absolute minimum when $x = 16$ since $A'(x) < 0$ for $0 < x < 16$ and $A'(x) > 0$ for $x > 16$. When $x = 16, y = 384/16 = 24$, so the dimensions are 24 cm and 36 cm.