<u>Substitution</u>

p. 467: 7-55 EOO, 61-71 odd, 75-79 odd

7. Let $u = x^3 + 1$. Then $du = 3x^2dx$ and $x^2dx = \frac{1}{3}du$,

so $\int x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\left(\frac{1}{3}du\right) = \frac{1}{3}\frac{u^{3/2}}{3/2} + C \;= \frac{2}{9}(x^3+1)^{3/2} + C.$

11. Let $u = 1 - x^2$. Then $du = -2xdx$ and $xdx = -\frac{1}{2}du$, so

$\int x\sqrt{1-x^2}\,dx = \int \sqrt{u}\left(-\frac{1}{2}du\right) = -\frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C \;= -\frac{1}{3}(1-x^2)^{3/2} + C.$

15. Let $u = \frac{\pi}{2}t$. Then $du = \frac{\pi}{2}dt$ and $dt = \frac{2}{\pi}du$, so $\int \cos\left(\frac{\pi}{2}t\right)dt = \int \cos u\left(\frac{2}{\pi}du\right) = \frac{2}{\pi}\sin u + C \;= \frac{2}{\pi}\sin(\frac{\pi}{2}t) + C.$

19. Let $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$ and $\sin\theta d\theta = -du$,

so $\int \cos^3\theta\sin\theta d\theta = \int -u^3\,du = -\frac{1}{4}u^4 + C \;= -\frac{1}{4}\cos^4\theta + C.$

23. Let $u = 3ax + bx^3$. Then $du = (3a + 3bx^2)\,dx = 3(a + bx^2)\,dx$, so

$\int \frac{a+bx^2}{\sqrt{3ax+bx^3}}dx = \int \frac{\frac{1}{3}du}{u^{1/2}} = \frac{1}{3}\int u^{-1/2}du = \frac{1}{3}\cdot 2u^{1/2} + C \;= \frac{2}{3}\sqrt{3ax+bx^3} + C.$

27. Let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$, so $\int \sec^2\theta\tan^3\theta\,d\theta = \int u^3\,du = \frac{1}{4}u^4 + C \;= \frac{1}{4}\tan^4\theta + C.$

31. Let $u = x^3 + 3x$. Then $du = (3x^2+3)dx$ and $\frac{1}{3}du = (x^2+1)dx$,

so $\int (x^2+1)(x^3+3x)^4\,dx = \int u^4\left(\frac{1}{3}du\right) = \frac{1}{3}\cdot\frac{1}{5}u^5 + C \;= \frac{1}{15}(x^3+3x)^5 + C.$

35. Let $u = \arctan x$. Then $du = \frac{1}{x^2+1}dx$ so $\int \frac{(\arctan x)^2}{x^2+1}dx = \int u^2du = \frac{1}{3}u^3 + C \;= \frac{1}{3}(\arctan x)^3 + C.$

39. Let $u = \cot x$. Then $du = -\csc^2 x\,dx$ and $\csc^2 x\,dx = -du$,

so $\int \sqrt{\cot x}\csc^2 x\,dx = \int -\sqrt{u}\,du = \frac{u^{3/2}}{3/2} + C = -\frac{2}{3}(\cot x)^{3/2} + C.$

43. $\int \frac{\sin 2x}{1+\cos^2 x}dx = 2\int \frac{\sin x\cos x}{1+\cos^2 x}dx = 2I$. Let $u = \cos x$. Then $du = -\sin x\,dx$, so

$2I = -2\int \frac{u\,du}{1+u^2} = -2\cdot\frac{1}{2}\ln(1+u^2) + C = -\ln(1+u^2) + C = -\ln\left(1+\cos^2 x\right) + C.$

47. Let $u = \sin^{-1}x$. Then $du = \frac{1}{\sqrt{1-x^2}}dx$ so $\int \frac{dx}{\sqrt{1-x^2}\sin^{-1}x} = \int \frac{du}{u} = \ln|u| + C \;= \ln\left|\sin^{-1}x\right| + C.$

51. Let $u = 2x + 5$. Then $du = 2\,dx$ and $x = \frac{1}{2}(u-5)$, so

$\int x(2x+5)^8\,dx = \int \frac{1}{2}(u-5)u^8\cdot\frac{1}{2}du = \frac{1}{4}\int (u^9 - 5u^8)\,du$

$= \frac{1}{4}\left(\frac{1}{10}u^{10} - \frac{5}{9}u^9\right) + C = \frac{1}{40}(2x+5)^{10} - \frac{5}{36}(2x+5)^9 + C$

55. Let $u = 2x + 1$. Then $du = 2\,dx$. When $x = 0, u = 1$, when $x = 4, u = 9$,

so $\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\int_1^9 u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big]_1^9 = \frac{1}{3}\left(9^{3/2}-1\right) = \frac{1}{3}\left(27-1\right) = \frac{26}{3}$, option (**C**).

61. Let $u = 3t - 1$, so $du = 3dt$. When $t = 0$, $u = -1$; when $t = 1, u = 2$. Thus,

$\int_0^1 (3t-1)^{50}\,dt = \int_{-1}^2 u^{50}\left(\frac{1}{3}du\right) = \frac{1}{3}\cdot\frac{1}{51}u^{51}\Big]_{-1}^2 = \frac{1}{153}\left(2^{51} - (-1)^{51}\right) = \frac{1}{153}(2^{51}+1)$

63. Let $u=5x+1$, so $du=5dx$. When $x=0$, $u=1$; when $x=3, u=16$. Thus,

$$\int_0^3 \frac{dx}{5x+1}=\frac{1}{5}\int_1^{16}\frac{1}{u}du=\frac{1}{5}\Big[\ln|u|\Big]_1^{16}=\frac{1}{5}(\ln 16-\ln 1)=\frac{1}{5}\ln 16.$$

65. Let $u=\frac{1}{2}t$, so $du=\frac{1}{2}dt$. When $t=\frac{\pi}{3}$, $u=\frac{\pi}{6}$; when $t=\frac{2\pi}{3}, u=\frac{\pi}{3}$. Thus,

$$\int_{\pi/3}^{2\pi/3}\csc^2\left(\tfrac{1}{2}t\right)dt=-\int_{\pi/6}^{\pi/3}\csc^2 u(2udu)=2(-\cot u)\Big]_{\pi/6}^{\pi/3}=-2\left(\cot\tfrac{\pi}{3}-\cot\tfrac{\pi}{6}\right)$$

$$=-2\left(\tfrac{1}{\sqrt{3}}-\sqrt{3}\right)=-2\left(\tfrac{1}{3}\sqrt{3}-\sqrt{3}\right)=\tfrac{4}{3}\sqrt{3}$$

67. Let $u=-x^2$, so $du=-2x\,dx$. When $x=0$, $u=0$; when $x=1, u=-1$. Thus,

$$\int_0^1 xe^{-x^2}\,dx=\int_0^{-1}-\tfrac{1}{2}e^u\,du=-\tfrac{1}{2}\Big[e^u\Big]_0^{-1}=-\tfrac{1}{2}\left(e^{-1}-e^0\right)=\tfrac{1}{2}(1-1/e).$$

69. Let $u=\sin x$, so $du=\cos x\,dx$. When $x=0$, $u=0$; when $x=\frac{\pi}{2}, u=1$. Thus,

$$\int_0^{\pi/2}\cos x\sin(\sin x)\,dx=\int_0^1\sin u\,du=-\cos u\Big]_0^1-(\cos 1-1)=1-\cos 1.$$

71. Assume $a>0$. Let $u=a^2-x^2$, so $du=-2x\,dx$. When $x=0$, $u=a^2$; when $x=a$, $u=0$. Thus,

$$\int_0^a x\sqrt{a^2-x^2}\,dx=\int_{a^2}^0-\tfrac{1}{2}u^{1/2}\,du=\tfrac{1}{2}\int_0^{a^2}u^{1/2}\,du=\tfrac{1}{2}\Big[\tfrac{2}{3}u^{3/2}\Big]_0^{a^2}=\tfrac{1}{3}a^3.$$

75. Let $u=1+2x$, so $x=\frac{1}{2}(u-1)$ and $du=2\,dx$. When $x=0$, $u=1$; when $x=4$, $u=9$. Thus,

$$\int_0^4\frac{x\,dx}{\sqrt{1+2x}}=\int_1^9\tfrac{1}{2}\frac{(u-1)}{\sqrt{u}}\frac{du}{2}=\tfrac{1}{4}\int_1^9\tfrac{1}{2}\left(u^{1/2}-u^{-1/2}\right)du=\tfrac{1}{4}\Big[\tfrac{2}{3}u^{3/2}-2u^{1/2}\Big]_1^9=\tfrac{1}{4}\cdot\tfrac{2}{3}\Big[u^{3/2}-3u^{1/2}\Big]_1^9$$

$$=\tfrac{1}{6}\Big[(27-9)-(1-3)\Big]=\tfrac{20}{6}=\tfrac{10}{3}.$$

77. Let $u=(x-1)^2$, so $du=2(x-1)\,dx$. When $x=0$, $u=1$; when $x=2, u=1$. Thus,

$\int_0^2(x-1)e^{(x-1)^2}dx=\int_1^1\tfrac{1}{2}e^u\,du=0$ since the limits are equal.

79. Let $u=\dfrac{2\pi t}{T}-\alpha$, so $du=u=\dfrac{2\pi}{T}dt$. When $t=0$, $u=-\alpha$; when $t=\dfrac{T}{2}, u=\pi-\alpha$. Thus,

$$\int_0^{T/2}\sin\left(\frac{2\pi t}{T}-\alpha\right)dt=-\frac{T}{2\pi}\int_{-\alpha}^{\pi-\alpha}\sin u\,du=\frac{T}{2\pi}[-\cos u]_{-\alpha}^{\pi-\alpha}=-\frac{T}{2\pi}[\cos(\pi-\alpha)-\cos(-\alpha)]$$

$$=-\frac{T}{2\pi}[-\cos\alpha-\cos\alpha]=-\frac{T}{2\pi}(-2\cos\alpha)=\frac{T}{\pi}\cos\alpha$$