Velocity and Tangents

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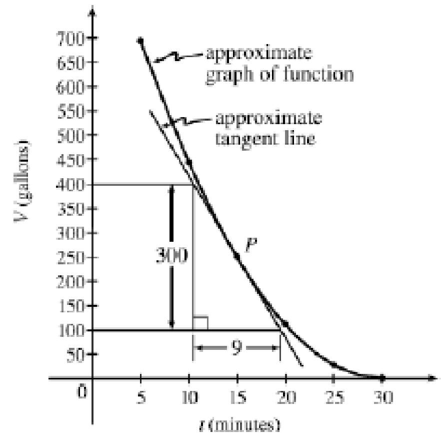
8. (a) Using $P = (15, 250)$ we construct the following table:

t	Q	slope = m_{PQ}
5	(5, 694)	$\frac{694-250}{5-15} = -\frac{444}{10} = -44.\overline{4}$
10	(10, 444)	$\frac{444-250}{10-15} = -\frac{194}{5} = -38.8$
20	(20, 111)	$\frac{111-250}{20-15} = -\frac{139}{5} = -27.8$
25	(25, 28)	$\frac{28-250}{25-15} = -\frac{272}{10} = -22.2$
30	(30, 0)	$\frac{0-250}{30-15} = -\frac{250}{15} =$ -16.7



(b) Using the values of t that correspond to the points closest to P ($t = 10$ and $t = 20$), we have $\frac{-33.8 + (-27.8)}{2} = -33.\overline{3}$.

(c) From the graph, we can estimate the slope of the tangent line at P to be $-33.\overline{3}$.

(d) The slope of the tangent line indicates that the volume of the water in the tank is decreasing at a rate of 33.3 gallons per minute at time $t = 15$ minutes.

13. (a) $y = y(t) = 10t - 1.86t^2$. At $t = 1, y = 10(1) - 1.86(1)^2 = 8.14$. The average velocity between times 1 and $1 + h$ is

$$v_{avg} = \frac{y(1+h) - y(1)}{(1+h) - 1} = \frac{\left[10(1+h) - 1.86(1+h)^2\right] - 8.14}{h} = \frac{6.28h - 1.86h^2}{h} = 6.28 - 1.86h, \text{ if } h \neq 0. \text{(i)}$$

$[1,2]$: $h = 1$, $v_{avg} = 4.42$ m/s (ii) $[1,1.5]$: $h = 0.5$, $v_{avg} = 5.35$ m/s

(iii) $[1,1.1]$: $h = 0.1$, $v_{avg} = 6.094$ m/s (iv) $[1,1.01]$: $h = 0.01$, $v_{avg} = 6.2614$ m/s

(v) $[1,1.001]$: $h = 0.001$, $v_{avg} = 6.27814$ m/s

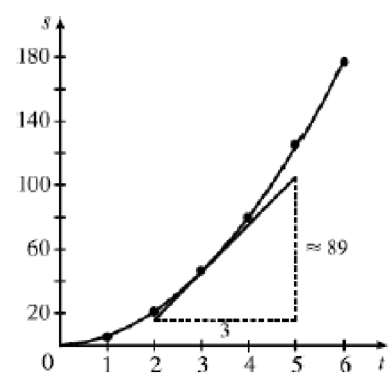
(b) The instantaneous velocity when $t = 1$ (h approaches 0) is 6.28 m / s.

14. (a) (i) On the interval [2, 4], $v_{avg} = \frac{s(4) - s(2)}{4 - 2} = \frac{79.2 - 20.6}{2} = 29.3$ ft/s.

(ii) On the interval [3, 4], $v_{avg} = \frac{s(4) - s(3)}{4 - 3} = \frac{79.2 - 46.5}{1} = 32.7$ ft/s.

(iii) On the interval [4, 5], $v_{avg} = \frac{s(5) - s(4)}{5 - 4} = \frac{124.8 - 79.2}{1} = 45.6$ ft/s.

(iv) On the interval [4, 6], $v_{avg} = \frac{s(6) - s(4)}{6 - 4} = \frac{176.7 - 79.2}{1} = 48.75$ ft/s.



(b) Using the points (2, 16) and (5, 105) from the approximate tangent line, the instantaneous velocity at $t = 3$ is about $\frac{105 - 16}{5 - 2} = \frac{89}{3} \approx 29.7$ ft/s.

15. (a) (i) $s = s(t) = 2\sin \pi t + 3\cos \pi t.$

On the interval [1, 2], $v_{avg} = \dfrac{s(2)-s(1)}{2-1} = \dfrac{3-(-3)}{1} = 6$ cm/s.

(ii) On the interval [1, 1.1], $v_{avg} = \dfrac{s(1.1)-s(1)}{1.1-1} = \dfrac{-3.471-(-3)}{0.1} = -4.71$ cm/s.

(iii) On the interval [1, 1.01], $v_{avg} = \dfrac{s(1.01)-s(1)}{1.01-1} = \dfrac{-3.0613-(-3)}{0.11} = -6.13$ cm/s.

(iv) On the interval [1, 1.001], $v_{avg} = \dfrac{s(1.001)-s(1)}{1.001-1} = \dfrac{-3.00627-(-3)}{0.001} = -6.27$ cm/s.

(b) The instantaneous velocity of the particle when $t = 1$ appears to be about – 6.3 cm/s.

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30. Let $s(t) = 40t - 16t^2$.

$$v(2) = \lim_{t\to 2}\frac{s(t)-s(2)}{t-2} = -\lim_{t\to 2}\frac{\left(40t-16t^2\right)-16}{t-2} = \lim_{t\to 2}\frac{-16t^2+40t-16}{t-2} = \lim_{t\to 2}\frac{-8\left(2t^2-5t+2\right)}{t-2}$$

$$= \lim_{t\to 2}\frac{-8(t-2)(2t-1)}{t-2} = -8\lim_{t\to 2}(2t-1) = -8(3) = -24.$$

Thus, the instantaneous velocity when $t = 2$ is –24 ft/s.

31. (a) Let $H(t) = 10t - 1.86t^2$.

$$v(1) = \lim_{h\to 0}\frac{H(1+h)-H(1)}{h} = \lim_{h\to 0}\frac{\left[10(1+h)-1.86(1+h)^2\right]-(10-1.86)}{h}$$

$$= \lim_{h\to 0}\frac{10+10h-1.86\left(1+2h+h^2-10+1.86\right)}{h}$$

$$= \lim_{h\to 0}\frac{10+10h-1.86-3.72h-1.86h^2-10+1.86}{h}$$

$$= \lim_{h\to 0}\frac{6.28h-1.86h^2}{h} = \lim_{h\to 0}(6.28-1.86h) = 6.28$$

The velocity of the rock after one second is 6.28 m/s.

(b) $$v(a) = \lim_{h\to 0}\frac{H(a+h)-H(a)}{h} = \lim_{h\to 0}\frac{\left[10(a+h)-1.86(a+h)^2\right]-(10a-1.86a^2)}{h}$$

$$= \lim_{h\to 0}\frac{10a+10h-1.86\left(1+2ah+h^2-10a+1.86a^2\right)}{h}$$

$$= \lim_{h\to 0}\frac{10a+10h-1.86a^2-3.72ah-1.86h^2-10a+1.86a^2}{h} = \lim_{h\to 0}\frac{10h-3.72ah-1.86h^2}{h}$$

$$= \lim_{h\to 0}\frac{h(10-3.72a-1.86h)}{h} = \lim_{h\to 0}(10-3.72a-1.86h) = 10-3.72a$$

The velocity of the rock when $t = a$ is $(10 - 3.72a)$ m/s.

(c) The rock will hit the surface when $H = 0$.

This is when $10t - 1.86t^2 = 0 \Leftrightarrow t(10-1.86t) = 0 \Leftrightarrow t = 0$ or $1.86t = 10$. The rock hits the surface when $t = 10/1.86 \approx 5.4$ s.

(d) The velocity of the rock when it hits the surface is $v\left(\frac{10}{1.86}\right) = 10 - 3.72\left(\frac{10}{1.86}\right) = 10 - 20 = -10$ m/s.

32. $v(a)=\lim_{h\to 0}\frac{s(a+h)-s(a)}{h}=\lim_{h\to 0}\frac{\frac{1}{(a+h)^2}-\frac{1}{a^2}}{h}=\lim_{h\to 0}\frac{\frac{a^2-(a+h)^2}{a^2(a+h)^2}}{h}=\lim_{h\to 0}\frac{a^2-\left(a^2+2ah+h^2\right)}{ha^2(a+h)^2}$

$=\lim_{h\to 0}\frac{-\left(2ah+h^2\right)}{ha^2(a+h)^2}=\lim_{h\to 0}\frac{-h(2a+h)}{ha^2(a+h)^2}=\lim_{h\to 0}\frac{-(2a+h)}{a^2(a+h)^2}=\frac{-2a}{a^2\cdot a^2}=-\frac{2}{a^3}$ m/s.

33. (a) The average velocity between times t and $t+h$ is

$\frac{s(t+h)-s(t)}{(t+h)-t}=\frac{\frac{1}{2}(t+h)^2-6(t+h)+23-\left(\frac{1}{2}t^2-6t+23\right)}{h}=\frac{\frac{1}{2}t^2+th+\frac{1}{2}h^2-6t-6h+23-\frac{1}{2}t^2+6t-23}{h}$

$=\frac{th+\frac{1}{2}h^2-6h}{h}=\frac{h\left(t+\frac{1}{2}h-6\right)}{h}=\left(t+\frac{1}{2}h-6\right)$ ft/s.

(i) [4, 8]: $t=4$, $h=8-4=4$, so the average velocity is $4+\frac{1}{2}(4)-6=0$ ft/s.

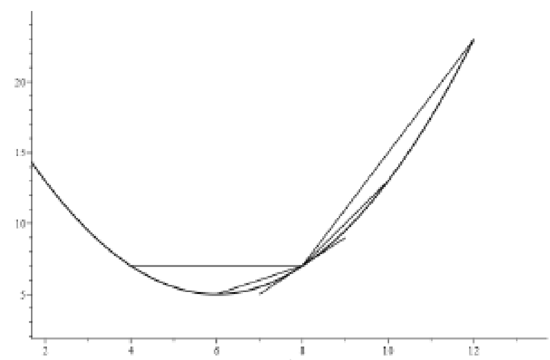
(ii) [6, 8]: $t=6$, $h=8-6=2$, so the average velocity is $6+\frac{1}{2}(2)-6=1$ ft/s.

(iii) [8, 10]: $t=8$, $h=10-8=2$, so the average velocity is $8+\frac{1}{2}(2)-6=3$ ft/s.

(iv) [8, 12]: $t=8$, $h=12-8=4$, so the average velocity is $8+\frac{1}{2}(4)-6=4$ ft/s.

(b) $v(t)=\lim_{h\to 0}\frac{s(t+h)-s(t)}{h}=\lim_{h\to 0}\left(t+\frac{1}{2}h-6\right)=t-6$, so $v(8)=2$ ft/s.

(c)



63. $v(4)=f'(4)=\lim_{h\to 0}\frac{f(4+h)-f(4)}{h}=\lim_{h\to 0}\frac{\left[80(4+h)-6(4+h)^2\right]-\left[80(4)-6(4)^2\right]}{h}$

$=\lim_{h\to 0}\frac{\left(320+80h-96-48h-6h^2\right)-(320-96)}{h}=\lim_{h\to 0}\frac{32h-6h^2}{h}$

$=\lim_{h\to 0}\frac{h(32-6h)}{h}=\lim_{h\to 0}(32-6h)=32$ m/s.

The speed when $t=4$ is $|32|=32$ m/s.