Parametric Equations

A curve can be described by the parametric equations

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

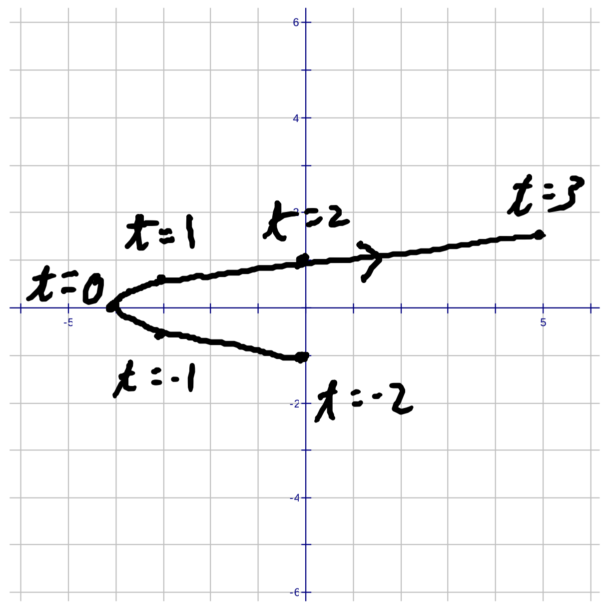
where x and y are functions of the parameter t.

→This allows us to describe a path as well as a direction along the path

→Think of t as the time at that location

Ex. Sketch $\begin{cases} x = t^2 - 4 \\ y = \frac{t}{2} \end{cases}$ for $-2 \le t \le 3$

t	$x = t^2 - 4$	$y = \frac{t}{2}$	
-2	0	-1	$(0, -1)$
-1	-3	$-\frac{1}{2}$	$(-3, -\frac{1}{2})$
0	-4	0	$(-4, 0)$
1	-3	$\frac{1}{2}$	$(-3, \frac{1}{2})$
2	0	1	$(0, 1)$
3	5	$\frac{3}{2}$	$(5, \frac{3}{2})$



Ex. Sketch $\begin{cases} x = \dfrac{1}{\sqrt{t+1}} \\ y = \dfrac{t}{t+1} \end{cases}$ for $t > -1$ by finding the Cartesian equation of the curve.

$\underline{t=0}$: $x = 1$, $y = 0$

$\underline{t=1}$: $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{2}$

$$x = \frac{1}{\sqrt{t+1}}$$

$$x^2 = \frac{1}{t+1}$$

$$\frac{1}{x^2} = t+1$$

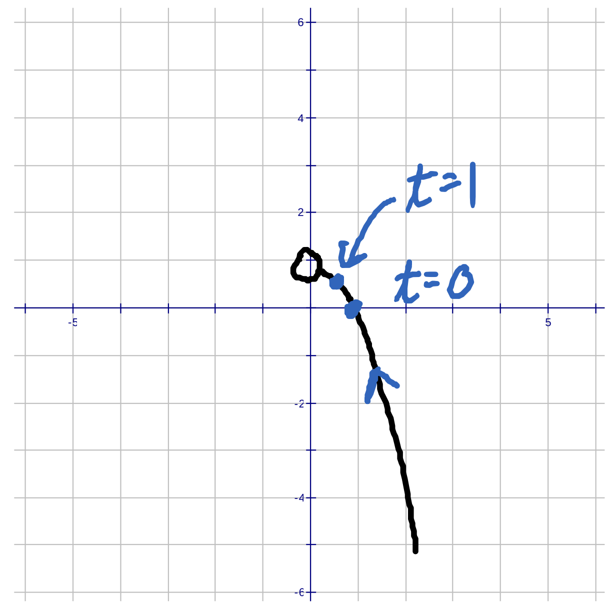
$$\frac{1}{x^2} - 1 = t$$

$$y = \frac{t}{t+1}$$

$$y = \frac{\frac{1}{x^2} - 1}{\frac{1}{x^2} - 1 + 1}$$

$$y = \frac{\left(\frac{1}{x^2} - 1\right) \cdot x^2}{\frac{1}{x^2} \cdot x^2} = 1 - x^2$$

$$y = 1 - x^2$$



Ex. Sketch $\begin{cases} x = 3\cos\theta \\ y = 4\sin\theta \end{cases}$ for $0 \le \theta \le 2\pi$

$\cos\theta = \frac{x}{3}$

$\sin\theta = \frac{y}{4}$

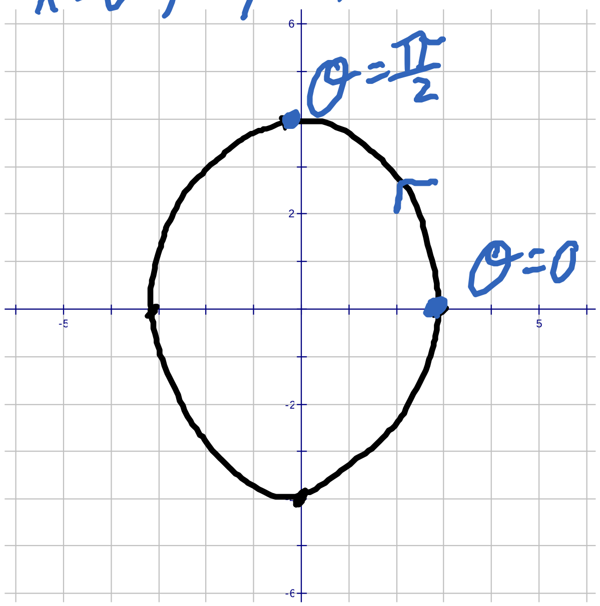
$\sin^2\theta + \cos^2\theta = 1$

$\left(\frac{y}{4}\right)^2 + \left(\frac{x}{3}\right)^2 = 1$

$\frac{x^2}{9} + \frac{y^2}{16} = 1$

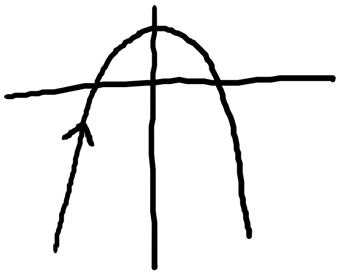
$\theta = 0: \ x = 3, \ y = 0$

$\theta = \frac{\pi}{2}: \ x = 0, \ y = 4$



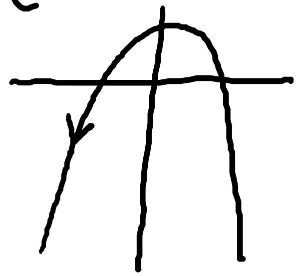
Ex. Express $y = 1 - x^2$ in multiple ways.

$$\begin{cases} x = t \\ y = 1 - t^2 \end{cases}$$



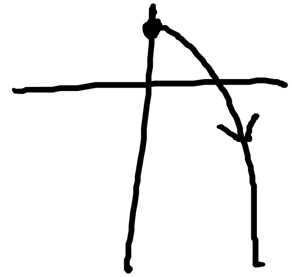
$(t, 1-t^2)$

$$\begin{cases} x = 2 - t \\ y = 1 - (2-t)^2 \end{cases}$$



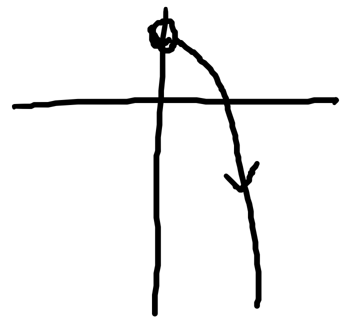
$$\begin{cases} x = \sqrt{t} \\ y = 1 - (\sqrt{t})^2 \end{cases}$$

$t \geq 0$



$(\sqrt{t}, 1-t)$

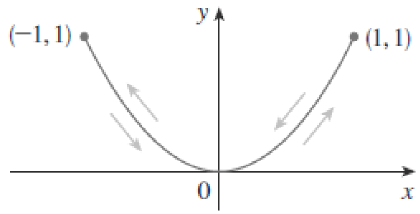
$$\begin{cases} x = e^t \\ y = 1 - (e^t)^2 \end{cases}$$



Parametric equations can also be written as vectors.

Pract.

1. Sketch the curve with equations $\begin{cases} x = \sin t \\ y = \sin^2 t \end{cases}$



2. Find the Cartesian equation of $\begin{cases} x = 1 + 3t \\ y = 2 - t^2 \end{cases}$

$$y = 2 - \left(\frac{x-1}{3}\right)^2$$

Calculus with Parametric Equations

If $x = f(t)$ and $y = g(t)$, then $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

<u>Ex.</u> Find $\dfrac{dy}{dx}$ for the curve given by $x = \sin t$ and $y = \cos t$, then find the equation of the line tangent to the curve at $t = \dfrac{\pi}{3}$.

$$\frac{dy}{dx} = \frac{-\sin t}{\cos t} \xrightarrow{\text{at } t = \frac{\pi}{3}}$$

$$m = \frac{-\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$$

$$x = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$y = \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\boxed{y - \frac{1}{2} = -\sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right)}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt}$$

<u>Ex.</u> Determine the slope and concavity of the curve given by $x = \sqrt{t}$, $y = \frac{1}{4}(t^2 - 4)$ at the point $(2,3)$

$\frac{dy}{dx} = \frac{\frac{1}{2}t}{\frac{1}{2}t^{-1/2}} = t^{3/2} \xrightarrow{t=4} \frac{dy}{dx} = 4^{3/2}$

$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\frac{3}{2}t^{1/2}}{\frac{1}{2}t^{-1/2}} = 3t \xrightarrow{t=4} \frac{d^2y}{dx^2} = 12$

$2 = \sqrt{t}$

$t = 4$

$3 = \frac{1}{4}(t^2 - 4)$

$12 = t^2 - 4$

$16 = t^2$

$t = \pm 4$

Arc length:

$$s = \int_a^b \sqrt{1 + (y')^2}\,dx = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Ex. Find the length of the curve given by $x = \ln t$, $y = t + 1$ for $1 \le t \le 6$

$$s = \int_1^6 \sqrt{\left(\frac{1}{t}\right)^2 + (1)^2}\, dt$$

Position: $(x(t), y(t))$

Velocity: $(x'(t), y'(t))$

Acceleration: $(x''(t), y''(t))$

Speed: $\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}$

Distance Traveled: $\int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$

<u>Ex.</u> The position of a particle is given by the vector $(5t^5 + \sin t, \ln t)$. Find the velocity vector, acceleration vector, and speed at any time t.

veloc. $= \left(25t^4 + \cos t, \ \frac{1}{t}\right)$

accel. $= \left(100t^3 - \sin t, \ \frac{-1}{t^2}\right)$

speed $= \sqrt{\left(25t^4 + \cos t\right)^2 + \left(\frac{1}{t}\right)^2}$

Ex. A particle moving along a curve has position $(x(t), y(t))$ at time $t \geq 0$ with

$$\frac{dx}{dt} = \sqrt{3t}, \qquad \frac{dy}{dt} = 3\cos\left(\frac{t^2}{2}\right)$$

The particle is at position $(1, 5)$ at time $t = 4$.

a) Find the acceleration vector at time $t = 4$.

$x''(4) = .433$

$y''(4) = -11.872$

accel $= (.433, -11.872)$

b) Find the y-coord. of the position at time $t = 0$.

$y(0) = ?$

$y(4) = 5$

$\int_0^4 y'(t)\,dt = y(4) - y(0)$

$y(0) = y(4) - \int_0^4 y'(t)\,dt = 1.601$

Ex. A particle moving along a curve has position $(x(t), y(t))$ at time $t \geq 0$ with

$$\frac{dx}{dt} = \sqrt{3t}, \qquad \frac{dy}{dt} = 3\cos\left(\frac{t^2}{2}\right)$$

The particle is at position $(1, 5)$ at time $t = 4$.

c) On the interval $0 \leq t \leq 4$, at what time does the speed first reach 3.5?

$$\sqrt{(x')^2 + (y')^2} = 3.5$$

$$t = 2.226$$

d) Find the total distance traveled over the interval $0 \leq t \leq 4$.

$$\int_0^4 \sqrt{(x')^2 + (y')^2}\, dt = 13.182$$

Circuit Time!