Calculus with Polar Coordinates

Ex. Graph all points of intersection of $r = 1 - 2\cos\theta$ and $r = 1$.

$\theta = 0$

$r = -1$

$\theta = \pi$

$1 - 2\cos\theta = 1$

$-2\cos\theta = 0$

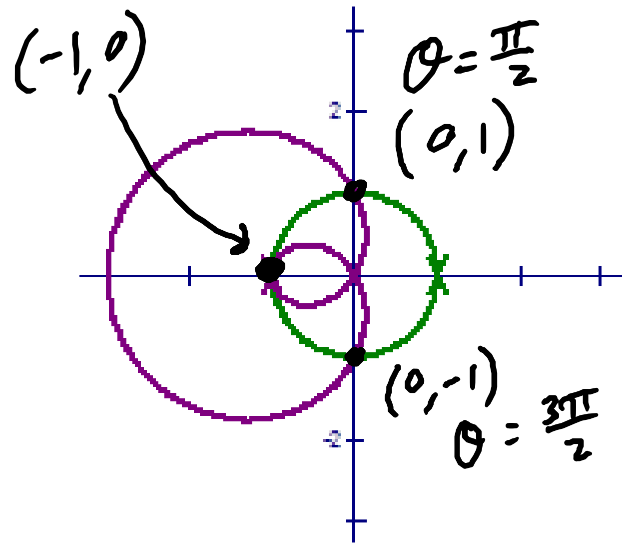
$\cos\theta = 0$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$x = r\cos\theta = 1\cos\frac{\pi}{2} = 0$

$y = r\sin\theta = 1\sin\frac{\pi}{2} = 1$

$(0,1)$ $(0,-1)$



The area bounded by $r = f(\theta), \alpha \leq \theta \leq \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta$$

$\underline{\text{Ex.}}$ Find the area of one petal of $r = 3\cos 3\theta$

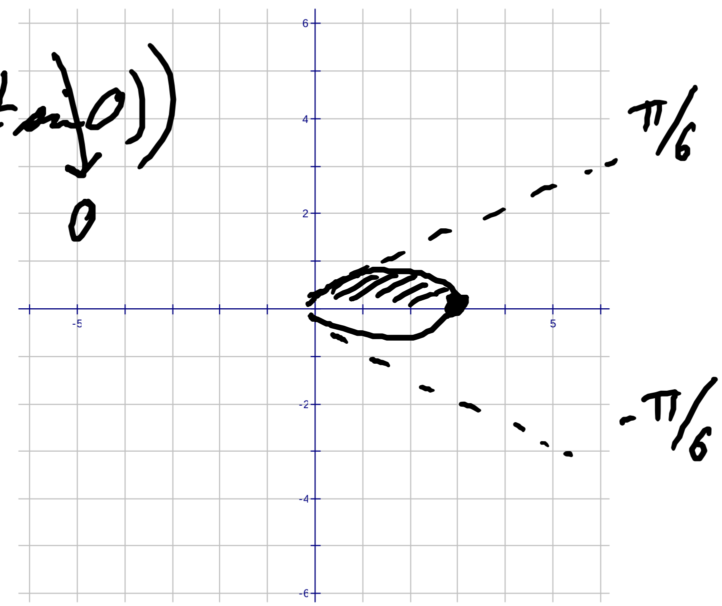
θ	$3\cos 3\theta$
0	$3\cos 0 = 3$
$\pi/6$	$3\cos\frac{\pi}{2} = 0$

$$A = 2\cdot\frac{1}{2}\int_0^{\pi/6}(3\cos 3\theta)^2\,d\theta = \int_0^{\pi/6} 9\cos^2(3\theta)\,d\theta$$

$$= \int_0^{\pi/6}\frac{9}{2}(1+\cos 6\theta)\,d\theta = \frac{9}{2}\left(\theta + \frac{1}{6}\sin 6\theta\right)\Big|_0^{\pi/6}$$

$$= \frac{9}{2}\left(\frac{\pi}{6} + \frac{1}{6}\underbrace{\sin\pi}_{0}\right) - \frac{9}{2}\left(0 + \frac{1}{6}\underbrace{\sin 0}_{0}\right)$$

$$= \boxed{\frac{3\pi}{4}}$$



Ex. Find the area between the inner and outer loops of the curve $r = 1 - 2\cos\theta$

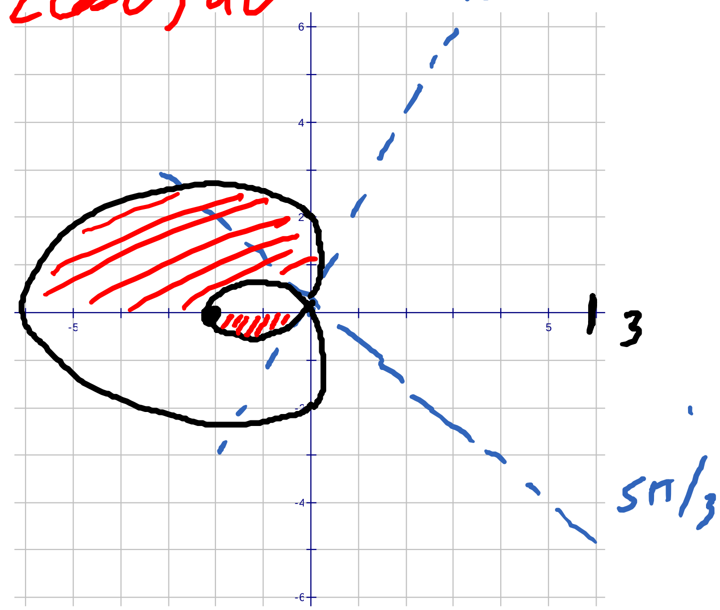
$$0 = 1 - 2\cos\theta$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \ \frac{5\pi}{3}$$

θ	$r = 1 - 2\cos\theta$
0	$1 - 2\cos 0 = -1$
$\pi/2$	$1 - 2\cos\frac{\pi}{2} = 1$
π	$1 - 2\cos\pi = 3$
$3\pi/2$	$1 - 2\cos\frac{3\pi}{2} = 1$
2π	$1 - 2\cos 2\pi = -1$

$\frac{\pi}{3}$ < 0

$\frac{5\pi}{3}$ < 0

$$A = 2\left[\frac{1}{2}\int_{\pi/3}^{\pi} (1 - 2\cos\theta)^2\, d\theta - \frac{1}{2}\int_{0}^{\pi/3} (1 - 2\cos\theta)^2\, d\theta\right]$$

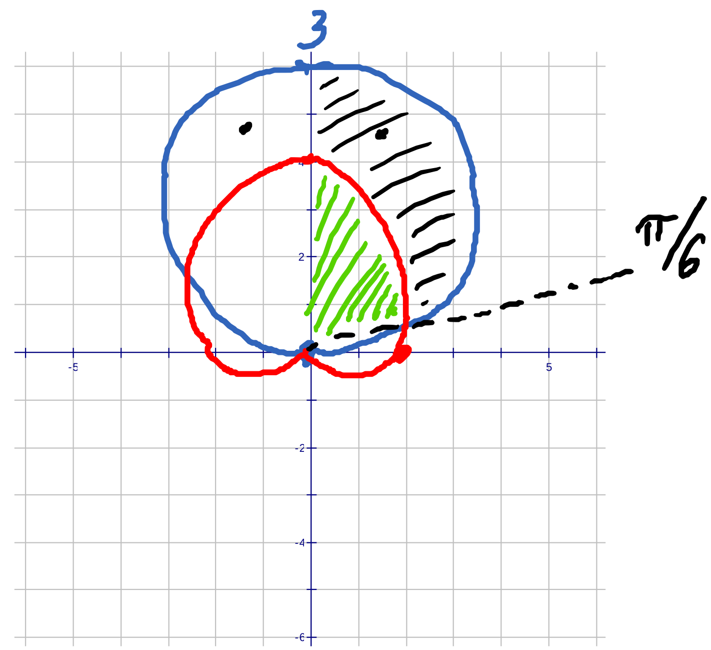


Ex. Find the area inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$.

θ	$3\sin\theta$	$1+\sin\theta$
0	$3\sin 0 = 0$	$1+\sin 0 = 1$
$\pi/2$	$3\sin\frac{\pi}{2} = 3$	$1+\sin\frac{\pi}{2} = 2$
π	$3\sin\pi = 0$	$1+\sin\pi = 1$
$3\pi/2$	$3\sin\frac{3\pi}{2} = -3$	$1+\sin\frac{3\pi}{2} = 0$
2π	$3\sin 2\pi = 0$	$1+\sin 2\pi = 1$

$$A = 2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2}(3\sin\theta)^2\,d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/2}(1+\sin\theta)^2\,d\theta\right]$$

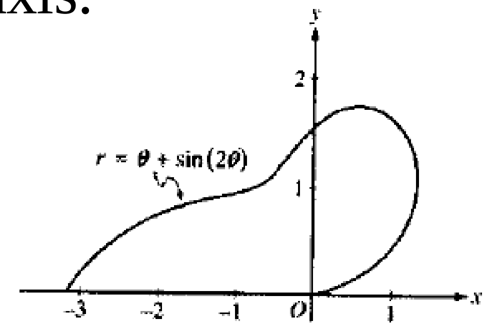
$$3\sin\theta = 1+\sin\theta$$
$$2\sin\theta = 1$$
$$\sin\theta = \frac{1}{2}$$
$$\theta = \pi/6$$



Ex. The curve is described by the equation $r = \theta + \sin 2\theta$, for $0 \leq \theta \leq \pi$.

a) Find the area bounded by the curve and the x-axis.

$$A = \frac{1}{2}\int_0^{\pi} (\theta + \sin 2\theta)^2 \, d\theta = 4.382$$



b) Find the angle θ that corresponds to the point with x-coordinate -2.

$$x = r\cos\theta = (\theta + \sin 2\theta)\cos\theta = -2$$

$$\theta = 2.786$$

$$r = \theta + \sin 2\theta$$

c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this say about the curve?

As θ increases, the graph gets closer to the origin

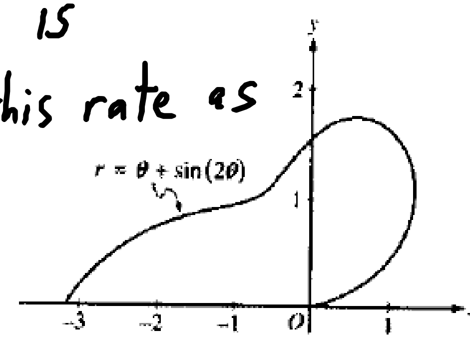
d) A particle is moving along the curve so that its position is $(x(t), y(t))$ and such that $\frac{d\theta}{dt} = 5$. Find $\frac{dy}{dt}$ at the instant that $\theta = \frac{\pi}{2}$ and interpret your answer.

$$y = (\theta + \sin 2\theta)\sin\theta$$

$$\frac{dy}{dt} = \frac{d}{d\theta}\left((\theta + \sin 2\theta)\sin\theta\right)\frac{d\theta}{dt}$$

$$\left.\frac{dy}{dt}\right|_{\theta = \frac{\pi}{2}} = -5$$

When $\theta = \frac{\pi}{2}$, y is decreasing at this rate as t increases



Matching Time!

Unit 9 Progress Check: MCQ Part A

- Do them all

Unit 9 Progress Check: MCQ Part B

- Do them all