Trigonometric Substitution

$$1 - \sin^2\theta = \cos^2\theta$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\sec^2\theta - 1 = \tan^2\theta$$

We will use these identities to change two terms (sum or difference of squares) into one term.

Ex. $\displaystyle\int \frac{1}{(x^2+1)^{3/2}}\,dx$

$$= \int \frac{1}{(\tan^2\theta + 1)^{3/2}} \sec^2\theta\, d\theta = \int \frac{1}{(\sec^2\theta)^{3/2}} \sec^2\theta\, d\theta$$

$$= \int \frac{1}{\sec^3\theta} \sec^2\theta\, d\theta = \int \frac{1}{\sec\theta}\, d\theta$$

$$= \int \cos\theta\, d\theta = \sin\theta + C$$

$$= \frac{x}{\sqrt{x^2+1}} + C$$

$x^2 + 1$

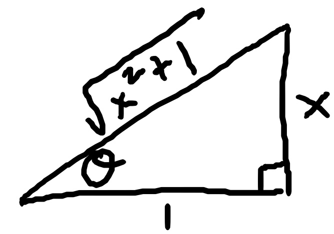
$\tan^2\theta + 1$

$\sec^2\theta$

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$\tan\theta = \frac{x}{1}$



Ex. $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}} = \int \frac{3\cos\theta\, d\theta}{9\sin^2\theta\sqrt{9-9\sin^2\theta}}$

$$= \int \frac{3\cos\theta\, d\theta}{9\sin^2\theta\sqrt{9(1-\sin^2\theta)}} = \int \frac{3\cos\theta\, d\theta}{9\sin^2\theta\sqrt{9\cos^2\theta}}$$

$$= \int \frac{3\cos\theta\, d\theta}{9\sin^2\theta \cdot 3\cos\theta} = \int \frac{1}{9\sin^2\theta}\, d\theta = \int \frac{1}{9}\csc^2\theta\, d\theta$$

$$= -\frac{1}{9}\cot\theta + C$$

$$= -\frac{1}{9}\,\frac{\sqrt{9-x^2}}{x} + C$$

$9 - x^2$

$9 - 9\sin^2\theta$

$9(1-\sin^2\theta)$

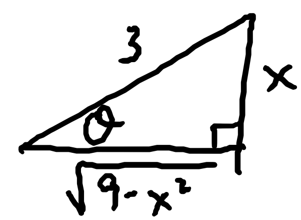
$9\cos^2\theta$

$x^2 = 9\sin^2\theta$

$x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$\sin\theta = \frac{x}{3}$



Ex. $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{4x^2+1}} = \int_0^{\pi/4} \frac{\frac{1}{2}\sec^2\theta\, d\theta}{\sqrt{\tan^2\theta+1}}$

$$= \int_0^{\pi/4} \frac{\frac{1}{2}\sec^2\theta}{\sec\theta}\, d\theta = \int_0^{\pi/4} \frac{1}{2}\sec\theta\, d\theta$$

$$= \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| \Big|_0^{\pi/4}$$

$$= \frac{1}{2}\ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \frac{1}{2}\ln\left|\sec 0 + \tan 0\right|$$

$$= \frac{1}{2}\ln\left(\sqrt{2} + 1\right)$$

$4x^2 + 1$

$\tan^2\theta + 1$

$\sec^2\theta$

$4x^2 = \tan^2\theta$

$2x = \tan\theta$

$x = \frac{1}{2}\tan\theta$

$dx = \frac{1}{2}\sec^2\theta\, d\theta$

$x = \frac{1}{2}$: $1 = \tan\theta \to \theta = \pi/4$

$x = 0$: $0 = \tan\theta \to \theta = 0$

Ex. $\int \frac{\sqrt{x^2-3}}{x}\,dx =$

$$= \int \frac{\sqrt{3\sec^2\theta-3}}{\sqrt{3}\sec\theta}\sqrt{3}\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\sqrt{3(\sec^2\theta-1)}}{\cancel{\sqrt{3}\sec\theta}}\cancel{\sqrt{3}\sec\theta}\tan\theta\,d\theta = \int \sqrt{3\tan^2\theta}\,\tan\theta\,d\theta$$

$$= \int \sqrt{3}\tan^2\theta\,d\theta = \int \sqrt{3}(\sec^2\theta-1)\,d\theta$$

$$= \sqrt{3}(\tan\theta - \theta) + C$$

$$= \sqrt{3}\left(\frac{\sqrt{x^2-3}}{\sqrt{3}} - \sec^{-1}\left(\frac{x}{\sqrt{3}}\right)\right) + C$$

x^2-3

$3\sec^2\theta-3$

$3(\sec^2\theta-1)$

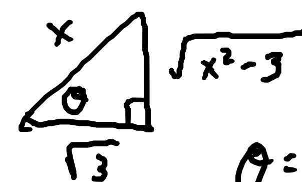
$3\tan^2\theta$

$x^2 = 3\sec^2\theta$

$x = \sqrt{3}\sec\theta$

$dx = \sqrt{3}\sec\theta\tan\theta\,d\theta$

$\sec\theta = \frac{x}{\sqrt{3}}$



$\theta = \sec^{-1}\frac{x}{\sqrt{3}}$

Ex. $\displaystyle\int \frac{x^2}{\sqrt{\cancel{2x-x^2}}}\,dx$

$1-(x-1)^2$

$$= \int \frac{(\sin\theta+1)^2}{\sqrt{\cos^2\theta}}\cos\theta\,d\theta = \int(\sin^2\theta+2\sin\theta+1)\,d\theta$$

$$= \int\left(\tfrac{1}{2}(1-\cos 2\theta)+2\sin\theta+1\right)d\theta$$

$$= \tfrac{1}{2}\left(\theta-\tfrac{1}{2}\underbrace{\sin 2\theta}_{2\sin\theta\cos\theta}\right)-2\cos\theta+\theta+C$$

$$= \tfrac{3}{2}\theta-\tfrac{1}{2}\sin\theta\cos\theta-2\cos\theta+C$$

$$= \tfrac{3}{2}\sin^{-1}(x-1)-\tfrac{1}{2}(x-1)\sqrt{2x-x^2}-2\sqrt{2x-x^2}+C$$

$1-(x-1)^2$

$1-\sin^2\theta$

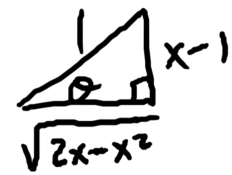
$\cos^2\theta$

$x-1=\sin\theta$

$x=\sin\theta+1$

$dx=\cos\theta\,d\theta$

$\sin\theta=\dfrac{x-1}{1}$



$\theta=\sin^{-1}(x-1)$

Pract.

1. $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$ $\quad \dfrac{-\sqrt{9-x^2}}{x} - \sin^{-1}\left(\dfrac{x}{3}\right) + c$

2. $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$ $\quad \dfrac{-\sqrt{x^2+4}}{4x} + c$

3. $\displaystyle\int \frac{x}{\sqrt{9x^2-1}}\,dx$ $\quad \dfrac{\sqrt{9x^2-1}}{9} + c$