# Rational Functions

A rational function looks like $\dfrac{P(x)}{Q(x)}$

To integrate, we will use partial fraction decomposition.

Ex. Decompose $\dfrac{5x+16}{x^2+6x+8} = \dfrac{3}{x+2} + \dfrac{2}{x+4}$

$(x+2)(x+4)$

$$\frac{5x+16}{(x+2)(x+4)} = \frac{A(x+4)}{(x+2)(x+4)} + \frac{B(x+2)}{(x+4)(x+2)}$$

$$5x+16 = A(x+4) + B(x+2)$$

$\underline{x=-2:}$ $\quad 6 = A(2) \longrightarrow A=3$

$\underline{x=-4:}$ $\quad -4 = B(-2) \longrightarrow B=2$

<u>Ex.</u> Decompose $\dfrac{3x-4}{x^2-2x} = \dfrac{2}{x} + \dfrac{1}{x-2}$

$x(x-2)$

$$\frac{3x-4}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

$$3x-4 = A(x-2) + Bx$$

$\underline{x=0:}$ $\quad -4 = A(-2) \longrightarrow A=2$

$\underline{x=2:}$ $\quad 2 = B(2) \longrightarrow B=1$

<u>Ex.</u> Decompose $\dfrac{-x+2}{x^2+2x+1} = \dfrac{-1}{x+1} + \dfrac{3}{(x+1)^2}$

$(x+1)^2$

$$\frac{-x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

$$-x+2 = A(x+1) + B$$

<u>$x=-1$</u>: $3 = B$

<u>$x=0$</u>: $2 = A(1) + 3$

$A = -1$

Degree 2 → $$\frac{}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

Degree 4 → $$\frac{}{(x+1)(x+2)^3} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}$$

Degree 3 → $$\frac{}{(x+1)(x^2+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2}$$

Degree 5 → $$\frac{}{(x+1)(x^2+2)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}$$

<u>Ex.</u> Decompose $\dfrac{8x+4}{x^4-1} = \dfrac{1}{x+1} + \dfrac{3}{x-1} + \dfrac{-4x-2}{x^2+1}$

$(x^2+1)(x^2-1)$

$(x+1)(x-1)$

$$\frac{8x+4}{(x^2+1)(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}$$

$$8x+4 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x+1)(x-1)$$

<u>$x=-1$:</u> $-4 = A(-2)(2) \rightarrow A=1$

<u>$x=1$:</u> $12 = B(2)(2) \rightarrow B=3$

<u>$x=0$:</u> $4 = 1(-1)(1) + 3(1)(1) + D(1)(-1) \longrightarrow 4=-1+3-D \rightarrow D=-2$

<u>$x=2$:</u> $20 = 1(1)(5) + 3(3)(5) + (2C-2)(3)(1) \longrightarrow 20 = 5+45+6C-6$

$-24 = 6C$

$C=-4$

Ex. $\int \frac{-x+2}{x^2+2x+1}\,dx = \int \left(\frac{3}{(x+1)^2} + \frac{-1}{x+1}\right) dx$

$= -3(x+1)^{-1} - \ln|x+1| + C$

**Ex.** $\displaystyle\int \frac{8x+4}{x^4-1}\,dx = \int \left(\frac{1}{x+1} + \frac{3}{x-1} + \frac{-4x-2}{x^2+1}\right) dx$

$$= \int\left(\frac{1}{x+1} + \frac{3}{x-1} + \frac{-4x}{x^2+1} + \frac{-2}{x^2+1}\right)dx$$

$$= \int\left(\frac{1}{x+1} + \frac{3}{x-1} - 4\,\frac{x}{x^2+1} - 2\,\frac{1}{x^2+1}\right)dx$$

$$= \ln|x+1| - 3\ln|x-1| - 2\ln|x^2+1| - 2\tan^{-1}x + C$$

$$\int \frac{x}{x^2+1}\,dx$$

$$\boxed{\begin{aligned} u &= x^2+1 \\ du &= 2x\,dx \\ \tfrac{1}{2}\,du &= x\,dx \end{aligned}}$$

$$= \int \frac{1}{u}\cdot\frac{1}{2}\,du$$

$$= \frac{1}{2}\ln|u|$$

$$= \frac{1}{2}\ln|x^2+1|$$

If the degree on top is not smaller than the bottom, divide.

Ex. $\displaystyle\int \frac{2x^3+7x^2+3x-5}{x^2+x}\,dx = \int \left(2x+5+\frac{-2x-5}{x^2+x}\right)dx$

$$\begin{array}{r} 2x+5+\dfrac{-2x-5}{x^2+x} \\ x^2+x\,\big)\,\overline{2x^3+7x^2+3x-5} \\ -(2x^3+2x^2) \\ \hline 5x^2+3x \\ -(5x^2+5x) \\ \hline -2x-5 \end{array}$$

$\displaystyle\int \left(2x+5+\frac{\quad}{x}+\frac{\quad}{x+1}\right)dx$

Pract.

1. $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$

$\frac{1}{2}\ln|x|+\frac{1}{10}\ln|2x-1|+c$

2. $\int \frac{x^3+x}{x-1}dx$ $\frac{1}{3}x^3+\frac{1}{2}x^2+2x+2\ln|x-1|+c$

- Inside function, derivative on outside

  →Substitution

- Product of two functions

  →Integration by Parts

- Powers of trig functions

  →Trig Integration

- Sum or difference of squares

  →Trig Substitution

- Rational Functions

  →Partial Fractions