Logistic Equations

Exponential functions can be used to model populations

→ These models are unbounded and can only be used for small populations over a short time span

→ In a real world situation, there are other factors that curtail this unbounded growth (food sources, space to grow, etc.)

→ A more realistic model is called a <u>logistic function</u>

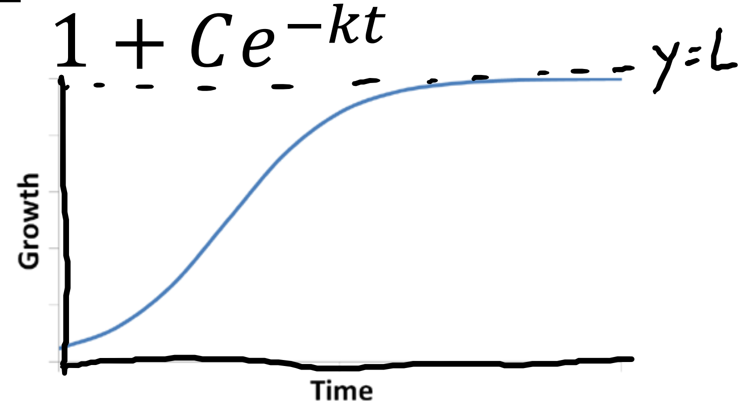
A growth model that features an upper and lower limit is defined by a logistic differential equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

The solution to this DE is a logistic function:

$$P = \frac{L}{1 + Ce^{-kt}}$$

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right) \qquad P = \frac{L}{1 + Ce^{-kt}}$$



L is the <u>carrying capacity</u> or <u>limiting value</u> – the value that the function approaches as $t \to \infty$

k is a constant that is part of the original DE

C is a constant that can be solved for using the initial value

Ex. In 2025, right after the onset of the zombie apocalypse, there were 15 reported cases of zombiism in San Diego. Two years later, there were 35 cases. The growth rate of the zombie population z is $\frac{dz}{dt} = kz\left(1 - \frac{z}{5000}\right)$ where t is years since 2025.

$z(0) = 15$
$z(2) = 35$

$L = 5000$

~~a)~~ Write a model for the zombie population in terms of t.

$z = \frac{5000}{1 + Ce^{-kt}}$

$15 = \frac{5000}{1 + Ce^{-k(0)}}$

$15 = \frac{5000}{1 + C}$

$1 + C = \frac{5000}{15}$

$C = 332.333$

$35 = \frac{5000}{1 + 332.333e^{-k(2)}}$

$1 + 332.333e^{-k(2)} = \frac{5000}{35}$

$332.333e^{-2k} = \frac{5000}{35} - 1$

$e^{-2k} = .427$

$-2k = -.851$

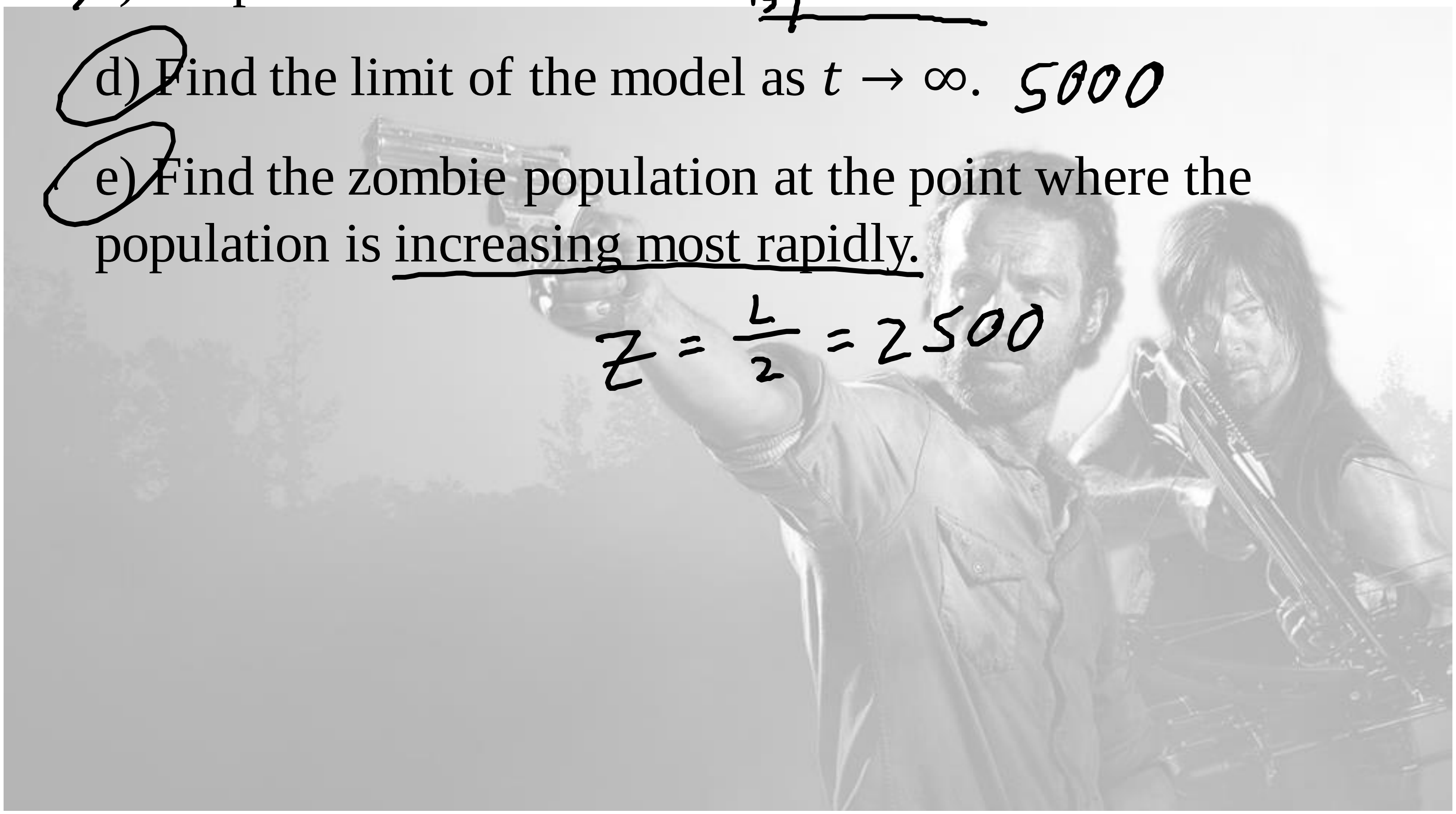
$k = .426$

$z = \frac{5000}{1 + 332.333e^{-.426t}}$

~~b)~~ Use the model to estimate the zombie population after 30 years.

$$Z = \frac{5000}{1+337.333e^{-.426(30)}} = 4995.276$$

~~c)~~ Graph the solution curve.



d) Find the limit of the model as $t \to \infty$. 5000

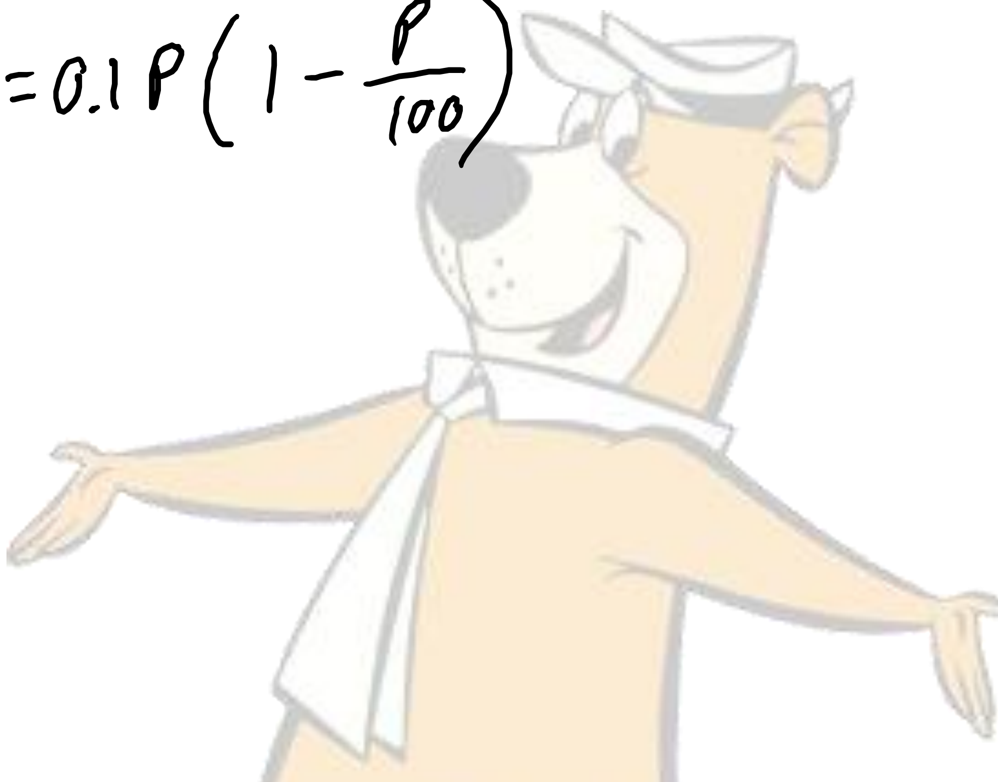
e) Find the zombie population at the point where the population is increasing most rapidly.

$$Z = \frac{L}{2} = 2500$$

Ex. Jellystone Park is capable of supporting no more than 100 bears. This can be modeled by a logistic differential equation with $k = 0.1$.

a) Write the differential equation.

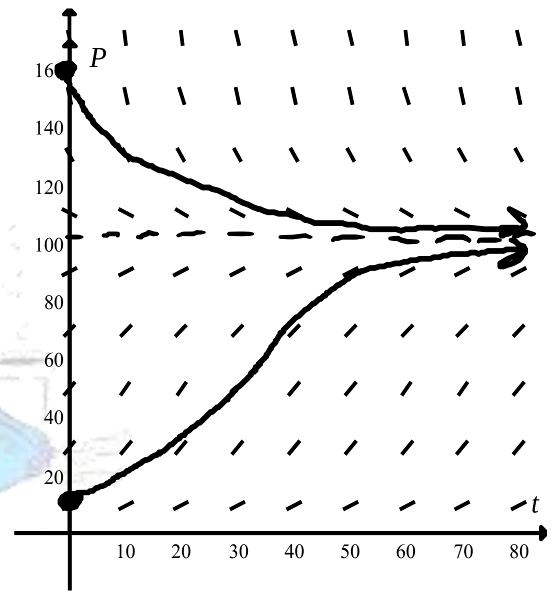
$$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{100}\right)$$



b) The slope field for this differential equation is shown. Where does there appear to be a horizontal asymptote? $P = 100$

What happens if the starting point is above the asymptote? Below?

c) If the park begins with 10 bears, sketch a graph of $P(t)$ on the slope field.



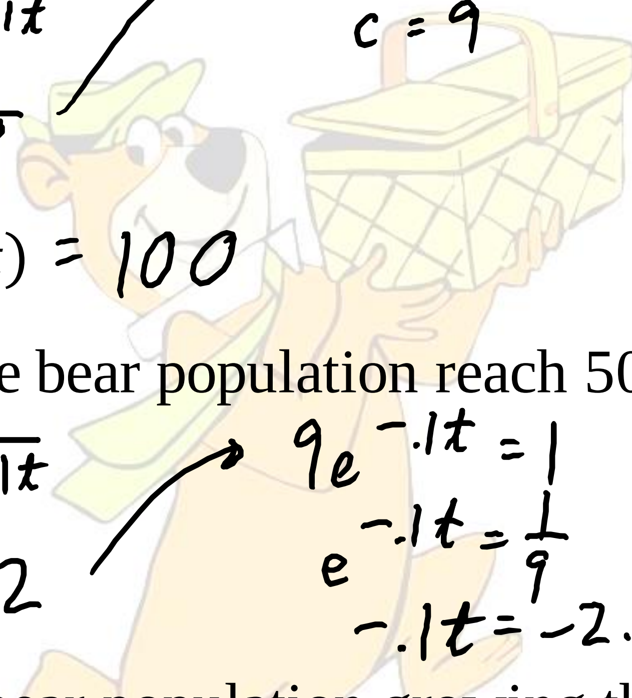
~~d)~~ Write the solution to the DE with initial condition $P(0) = 10$

$$P = \frac{100}{1 + Ce^{-.1t}}$$

$$10 = \frac{100}{1 + Ce^{0}} \rightarrow 1 + C = 10$$

$$C = 9$$

$$P = \frac{100}{1 + 9e^{-.1t}}$$



e) Find $\lim_{t\to\infty} P(t) = 100$

~~f)~~ When will the bear population reach 50?

$$50 = \frac{100}{1 + 9e^{-.1t}}$$

$$1 + 9e^{-.1t} = 2 \rightarrow 9e^{-.1t} = 1$$

$$e^{-.1t} = \frac{1}{9}$$

$$-.1t = -2.197 \rightarrow t = 21.972$$

~~g)~~ When is the bear population growing the fastest? $\rightarrow t = 21.972$

What is Pop. when growing fastest? $\rightarrow 50$

Unit 7 Progress Check: MCQ Part A

- Do #4-12

Unit 7 Progress Check: MCQ Part B

- Do #4-6, 10-12