Warm-up Problems

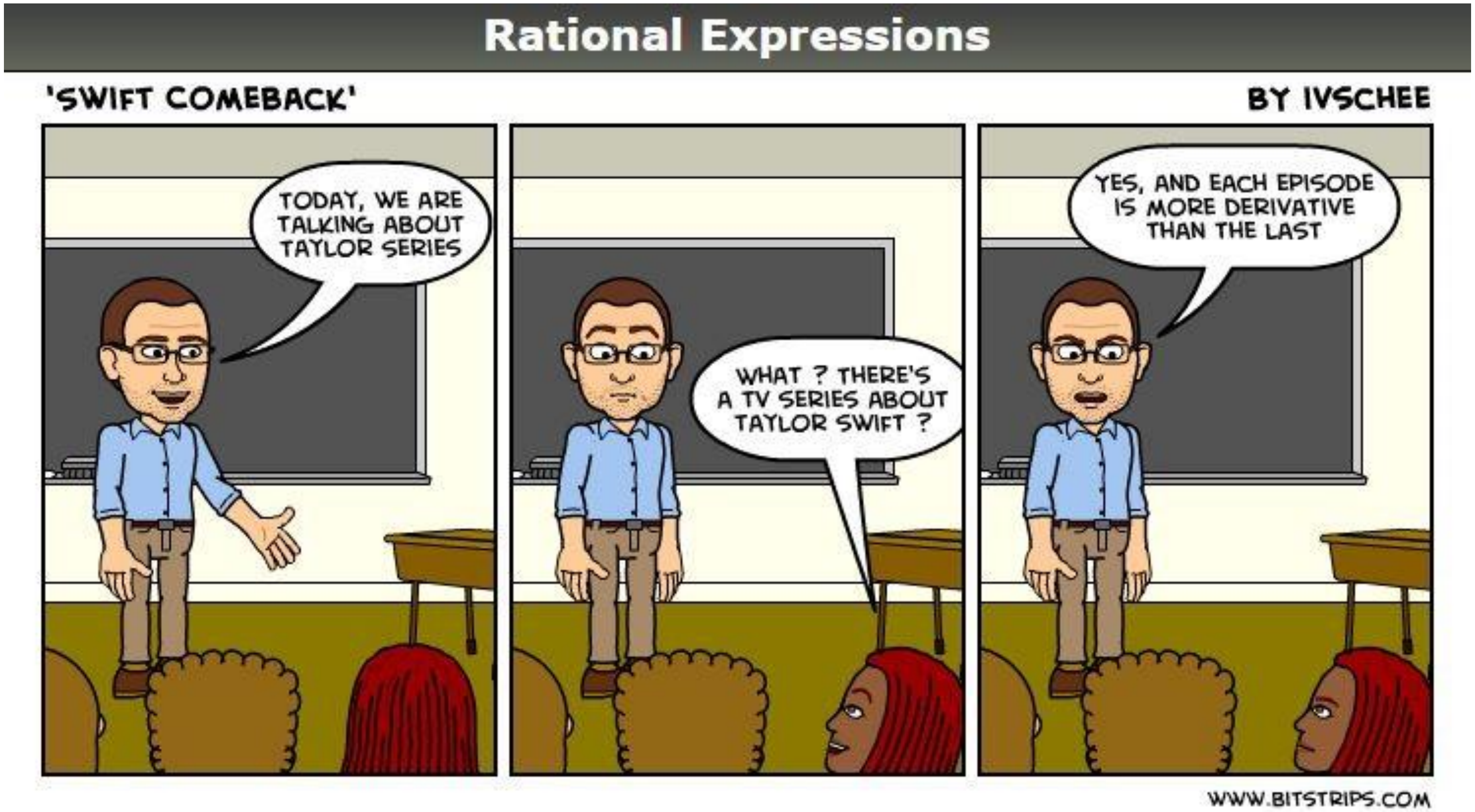
Write each series as a function.

1) $x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \mathrm{L}$

2) $1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \mathrm{L}$

3) $\frac{x-1}{x} - \frac{(x-1)^2}{2x} + \frac{(x-1)^3}{3x} - \frac{(x-1)^4}{4x} + \mathrm{L}$

4) $1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \mathrm{L}$


Rational Expressions
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BY IVSCHEE
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A Taylor polynomial considers only the first few terms of a Taylor series and can be used to approximate the value of the function.

Ex. Let f be a function such that $f(2) = 4$, $f'(2) = -1$, and $f''(2) = 15$. Find T_2, the second-degree Taylor polynomial centered at $x = 2$. Approximate $f(1)$.

$$T_2(x) = f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2$$

$$= 4 - (x-2) + \frac{15}{2}(x-2)^2$$

$$f(1) \approx 4 - (-1) + \frac{15}{2}(-1)^2 = 12.5$$

Ex. Let $P_4(x) = 15 - 3x + 6x^2 - 14x^3 - 7x^4$ be the Taylor polynomial for $f(x)$ about $x = 0$. Find $f'''(0)$.

$$-14 = \frac{f'''(0)}{3!}$$

$$f'''(0) = (-14)(3!)$$

$$= -84$$

Ex. $P(x) = -\frac{1}{2!} + \frac{1}{4!}x^2 - \frac{1}{6!}x^4$ is the Taylor polynomial for $f(x)$ centered at $x = 0$. Determine if f has a local max., local min, or neither at $x = 0$.

crit pt. at $x=0$? $\Rightarrow$ $f'(0) \overset{?}{=} 0$ yes, x^1 term is missing

$f''(0) > 0 \Rightarrow$ concave up

$\Rightarrow x=0$ local min.

Error in Taylor Series

Thm. Taylor's Theorem (Lagrange Error)

Suppose $P_n(x)$, the n-th degree Taylor polynomial for $f(x)$ centered at a, is used to approximate the value of f on a closed interval I containing a. Then

$$\boxed{\left| f(x) - P_n(x) \right|} \le \frac{\max\left(\left| f^{(n+1)} \right| \text{ on } I\right)}{(n+1)!} \boxed{\left| x - a \right|}^{n+1}$$

for all x in I.

Error

Distance from center

Ex. Write the second degree Maclaurin polynomial, $M_2(x)$, for $f(x) = e^x$ and use it to approximate $e^{0.9}$. Find the maximum error in using this approximation.

$$f(x) \approx 1 + x + \frac{1}{2}x^2$$

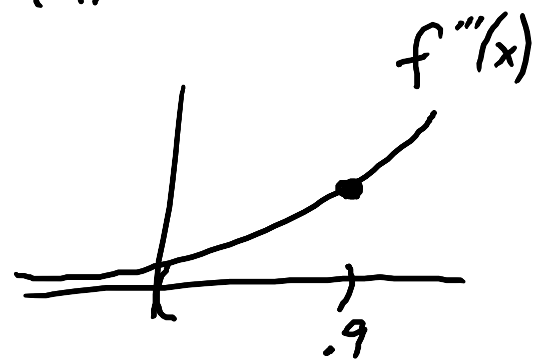
$$f(.9) = 1 + .9 + \frac{1}{2}(.9)^2$$

$$\text{error} \le \frac{\max(f''')}{3!}(.9)^3$$

$$\le \frac{e^{.9}}{3!}(.9)^3$$

$$\le .299$$

$f'''(x) = e^x$



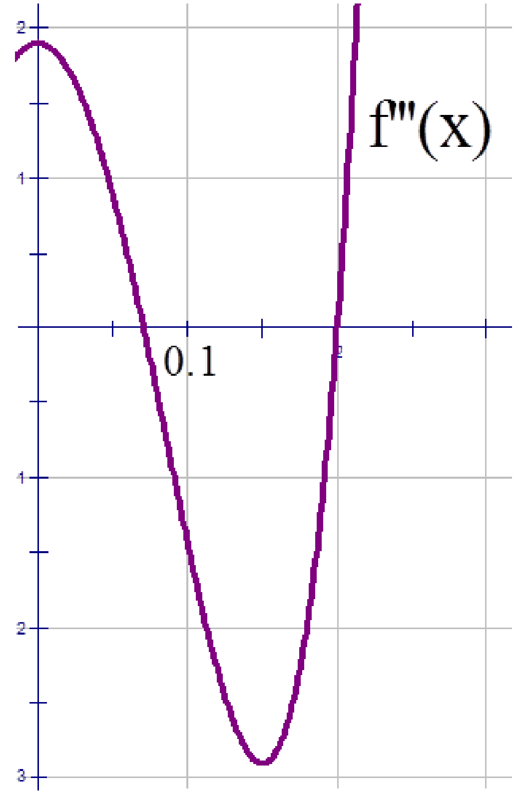
Ex. Assume the 2nd degree Maclaurin polynomial for $f(x)$, $P_2(x) = 1 + x - \frac{1}{3}x^2$, is used to approximate $f(.2)$. Find a possible range of values for $f(.2)$.

$$\text{error} \le \frac{\max|f'''|}{3!}(.2)^3$$

$$\le \frac{3}{3!}(.2)^3$$

$$f(.2) \approx P_2(.2) = 1 + .2 - \frac{1}{3}(.2)^2$$

$$\underbrace{1 + .2 - \tfrac{1}{3}(.2)^2}_{\text{approx}} - \underbrace{\tfrac{3}{3!}(.2)^3}_{\text{error}} \le f(.2) \le \underbrace{1 + .2 - \tfrac{1}{3}(.2)^2}_{\text{approx}} + \underbrace{\tfrac{3}{3!}(.2)^3}_{\text{error}}$$

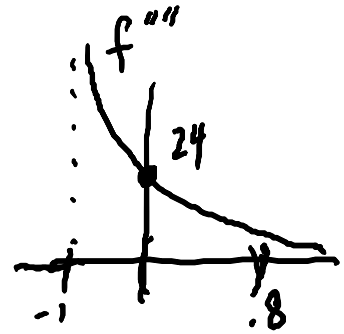


Pract. Write the 3rd degree Maclaurin polynomial for

$f(x)=\dfrac{1}{1+x}$ and use it to approximate $\dfrac{1}{1+.8}$. Find the maximum error in using this approximation.

$$f(x)=\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}(-x)^n=\sum_{n=0}^{\infty}(-1)^n x^n=1-x+x^2-x^3+\dots$$

$$f(.8)\approx 1-.8+(.8)^2-(.8)^3$$



$$f(x)=(1+x)^{-1}$$
$$f'(x)=-(x+1)^{-2}$$
$$f''(x)=2(x+1)^{-3}$$
$$f'''(x)=-6(x+1)^{-4}$$
$$f''''(x)=24(x+1)^{-5}$$

$$\text{error}\le\frac{\max|f''''|}{4!}(.8)^4$$

$$\le\frac{24}{4!}(.8)^4$$

This approximation is an alternating series, so we could consider the error from that direction…

Ex. Let $f(x) = \sin(x^2) + \cos x$.

a) Write the first four nonzero terms of the Maclaurin series for

i) $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots$

ii) $\sin(x^2) = x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} - \frac{1}{7!}x^{14} + \ldots$

iii) $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots$

iv) $f(x) = 1 + \left(1 - \frac{1}{2!}\right)x^2 + \frac{1}{4!}x^4 + \left(-\frac{1}{3!} - \frac{1}{6!}\right)x^6 + \ldots$

$= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{121}{720}x^6 + \ldots$

$\underline{\text{Ex.}}$ Let $f(x) = \sin(x^2) + \cos x$.

$$f(x) \approx 1 + \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \tfrac{121}{720}x^6 + \ldots$$

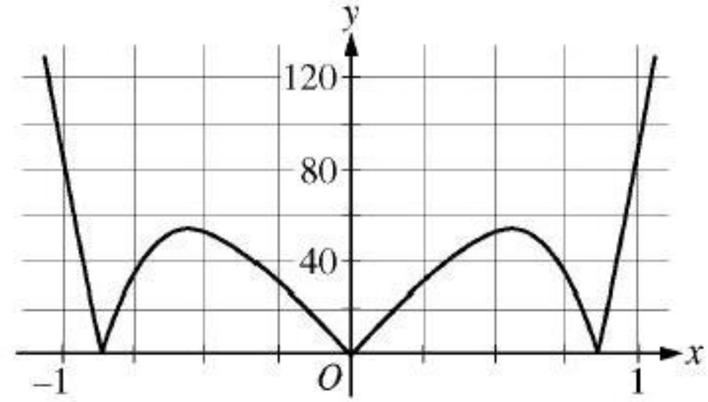
b) Find the value of $f^{(6)}(0)$.

$$-\frac{121}{720} = \frac{f^{(6)}(0)}{6!}$$

$$f^{(6)}(0) = -121$$

c) The graph of $|f^{(5)}(x)|$ is shown below. Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Show that

$$\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$$



Graph of $y = \left|f^{(5)}(x)\right|$

$$\text{error} \le \frac{\max\left|f^{(5)}\right|}{5!}\left(\frac{1}{4}\right)^5$$

$$\le \frac{40}{5!\,4^5} \le \frac{1}{3072} < \frac{1}{3000} \checkmark$$

Unit 10 Progress Check: MCQ Part A

- Do them all

Unit 10 Progress Check: MCQ Part B

- Do them all

Unit 10 Progress Check: MCQ Part C

- Do them all