Taylor and Maclaurin Series

Last class, we found the power series for some specific forms of functions. Today we'll find the power series for any function $f(x)$.

Thm. If f has a power series representation centered at a, meaning that f can be written

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

then the coefficients will be $c_n = \dfrac{f^{(n)}(a)}{n!}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

This is called the <u>Taylor series</u> representation of $f(x)$ centered at $x = a$.

When $a = 0$, we call it the Maclaurin series.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \cdots$$

Ex. Find the first 4 nonzero terms of the Taylor series for $f(x) = \ln x$ centered at $x = 1$, then write the series in sigma notation.

$f(x) = \ln x \to f(1) = 0$

$f'(x) = \frac{1}{x} = x^{-1} \to f'(1) = 1$

$f''(x) = -x^{-2} \to f''(1) = -1$

$f'''(x) = 2x^{-3} \to f'''(1) = 2$

$f^{(4)}(x) = -6x^{-4} \to f^{(4)}(1) = -6$

$$f(x) = f(1) + \frac{f'(1)}{1!}(x-1)^1 + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \dots$$

$$= 0 + \frac{1}{1}(x-1)^1 - \frac{1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 + \frac{-6}{24}(x-1)^4 + \dots$$

$$= \underbrace{(x-1)}_{n=1} - \underbrace{\frac{1}{2}(x-1)^2}_{n=2} + \underbrace{\frac{1}{3}(x-1)^3}_{n=3} - \underbrace{\frac{1}{4}(x-1)^4}_{n=4} + \dots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

Ex. Find the first 4 nonzero terms of the Maclaurin series for $f(x) = \sin x$, then write the series in sigma notation.

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$

$f^{(5)}(x) = \cos x \qquad f^{(5)}(0) = 1$

$f^{(6)}(x) = -\sin x \qquad f^{(6)}(0) = 0$

$f^{(7)}(x) = -\cos x \qquad f^{(7)}(0) = -1$

$$f(x) = f(0) + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \frac{f^{(6)}}{6!}x^6 + \frac{f^{(7)}}{7!}x^7 + \cdots$$

$$= \underbrace{\frac{1}{1!}x^1}_{n=0} + \underbrace{\frac{-1}{3!}x^3}_{n=1} + \underbrace{\frac{1}{5!}x^5}_{n=2} + \underbrace{\frac{-1}{7!}x^7}_{n=3} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Ex. Find the first 4 nonzero terms of the Maclaurin series for $f(x) = e^x$, then write the series in sigma notation.

$$f(x) = e^x \qquad f(0) = 1$$
$$f'(x) = e^x \qquad f'(0) = 1$$
$$f''(x) = e^x \qquad f''(0) = 1$$
$$f'''(x) = e^x \qquad f'''(0) = 1$$

$$f(x) = f(0) + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= \underbrace{1}_{n=0} + \underbrace{x}_{n=1} + \underbrace{\frac{1}{2!}x^2}_{n=2} + \underbrace{\frac{1}{3!}x^3}_{n=3} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

We can use these to expand our functions

Ex. Find the first 4 non-zero terms and the general term for the Maclaurin series for $f(x) = \sin x^2$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \dots$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots + \frac{(-1)^n}{(2n+1)!} x^{4n+2} + \dots$$

$$(x^2)^{2n+1} = x^{2(2n+1)}$$

Ex. Find the first 4 non-zero terms and the general term for the Maclaurin series for $f(x) = x\cos\sqrt{x}$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n}{(2n)!}x^{2n} + \dots$$

$$\cos\sqrt{x} = 1 - \frac{x^1}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \dots + \frac{(-1)^n}{(2n)!}x^{n} + \dots$$

$$x\cos\sqrt{x} = x - \frac{x^2}{2!} + \frac{x^3}{4!} - \frac{x^4}{6!} + \dots + \frac{(-1)^n}{(2n)!}x^{n+1} + \dots$$

Ex. Find the first three non-zero terms of the Maclaurin series for $\int e^{-x^2}dx$

$$e^x = 1 + x + \frac{1}{2}x^2 + \ldots$$

$$e^{-x^2} = 1 - x^2 + \frac{1}{2}x^4 + \ldots$$

$$\int e^{-x^2}dx = C + x - \frac{1}{3}x^3 + \frac{1}{10}x^5 + \ldots$$

Ex. Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{6}\right)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

Ex. Find a power series for $f(x)$ centered at $a = 1$ if $f(1) = 2$ and $f^{(n)}(1) = n!$.

$f(1) = 2$

$f'(1) = 1!$

$f''(1) = 2!$

$f'''(1) = 3!$

$\vdots$

$$f(x) = \cancel{2+} \frac{\cancel{1!}}{\cancel{1!}}(x-1)^1 + \frac{\cancel{2!}}{\cancel{2!}}(x-1)^2 + \frac{\cancel{3!}}{\cancel{3!}}(x-1)^3 + \dots$$

$1 + 1 +$

$n=1$ $\quad n=2$ $\quad n=3$

$$= 2 + \sum_{n=1}^{\infty} (x-1)^n$$

A Taylor polynomial considers only the first few terms of a Taylor series and can be used to approximate the value of the function.

Ex. Let f be a function such that $f(2) = 4$, $f'(2) = -1$, and $f''(2) = 15$. Find T_2, the second-degree Taylor polynomial centered at $x = 2$. Approximate $f(1)$.

$$T_2(x) = f(2) + \frac{f'(2)}{1!}(x-2)^1 + \frac{f''(2)}{2!}(x-2)^2$$

$$= 4 + \frac{-1}{1}(x-2) + \frac{15}{2}(x-2)^2$$

$$f(1) \approx T_2(1) = 4 - (-1) + \frac{15}{2}(-1)^2$$

Ex. Let $P_4(x) = 15 - 3x + 6x^2 - 14x^3 - 7x^4$ be the Taylor polynomial for $f(x)$ about $x = 0$. Find $f'''(0)$.

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}$$

$$\frac{f'''(0)}{3!} = -14$$

$$f'''(0) = (-14)(6)$$

Ex. $P(x) = -\frac{1}{2!} + \frac{1}{4!}x^2 - \frac{1}{6!}x^4$ is the Taylor polynomial for $f(x)$ centered at $x = 0$. Determine if f has a local max., local min, or neither at $x = 0$.

$0x^1$

$f'(0) \stackrel{?}{=} 0$

~~local max → f' pos to neg~~

~~local min → f' neg to pos.~~

$\frac{f'(0)}{1!} = 0$

$f'(0) = 0$ ✓

$x=0$ is crit. pt.

local max → $f''(0) < 0$

local min → $f''(0) > 0$

$\frac{f''(0)}{2!} = \frac{1}{4!}$

$f''(0) = \frac{2!}{4!} > 0$ → $\boxed{x=0 \text{ is local min}}$