Integral Test

So far, the Test for Divergence tells us if a series diverges and the Geometric Series Test tells us about the convergence of those series.

→ Over the next few lessons, we will learn several more ways to determine the convergence of a series.

→ When citing the name of a test as justification, abbreviate at your own risk

Thm. Integral Test

Let f be a positive, decreasing, continuous function for $x \geq 1$ such that $f(n) = a_n$. Then

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)dx$$

either both converge or both diverge.

$\underline{\text{Ex.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$f(x) = \frac{x}{x^2+1}$ pos. ✓

$f'(x) = \frac{(x^2+1)\cdot 1 - x\cdot 2x}{(x^2+1)^2}$ cont. ✓

dec. ✓

$= \frac{1-x^2}{(x^2+1)^2} < 0$

$$\int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{R\to\infty}\int_1^R \frac{x}{x^2+1}\,dx$$

$$= \lim_{R\to\infty} \frac{1}{2}\ln|x^2+1|\Big|_1^R = \lim_{R\to\infty} \frac{1}{2}\ln|R^2+1| - \frac{1}{2}\ln|2| = \infty$$

$\frac{1}{2}\ln|R^2+1| \to \infty$

div.

$\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ div. by Integral Test.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$ is called a <u>p-series</u>.

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$$ is called the <u>harmonic series</u>.

<u>Thm.</u> p-Series Test

The p-series converges if $p > 1$ and diverges if $p \leq 1$.

Ex. Determine the convergence of

a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$ conv. by p-series test, $p = 3$

b) $\sum_{n=1}^{\infty} \frac{1}{n}$ div. by harm. series

c) $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ conv.

$n=1$ $\quad n=2 \to 2 \cdot 2^{1/2}$ $\quad n=3 \to 3 \cdot 3^{1/2}$ $\quad n=4$ $\quad n \cdot n^{1/2}$

Ex. Determine the convergence of

a) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ $\qquad \sum \frac{1}{\sqrt{n}}$ div.

b) $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ $\qquad \sum \frac{1}{n}$ div.

c) $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ $\qquad \sum \frac{1}{n^3}$ conv.

Comparison Tests

<u>Thm.</u> Limit Comparison Test

Consider $a_n > 0$ and $b_n > 0$, and suppose there is a finite positive L such that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$

Then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$ $\xrightarrow{\text{compare}}$ $\displaystyle\sum \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{\frac{1}{2+\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2+\sqrt{n}} = 1$$

$\sum \frac{1}{\sqrt{n}}$ div. by p-series test, $p = \frac{1}{2}$

$\therefore$ $\sum \frac{1}{2+\sqrt{n}}$ div. by Limit Comp. Test

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ $\longrightarrow$ $\displaystyle\sum \frac{\sqrt{n}}{n^2} = \sum \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{n^2+1} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$

$\sum \frac{1}{n^{3/2}}$ conv. by p-series test, $p = \frac{3}{2}$

$\therefore \sum \frac{\sqrt{n}}{n^2+1}$ conv. by Limit comp. test

$$\underline{\text{Ex.}} \sum_{n=1}^{\infty} \frac{\sqrt{n}}{3n^2 - 4n + 5} \longrightarrow \sum \frac{\sqrt{n}}{n^2} = \sum \frac{1}{n^{3/2}}$$

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{3n^2-4n+5}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^2}{3n^2-4n+5} = \frac{1}{3}$$

$\therefore \sum \frac{1}{n^{3/2}}$ conv. by p-series test, $p = \frac{3}{2}$

$\therefore \sum \frac{\sqrt{n}}{3n^2-4n+5}$ conv. by limit comp. test.

<u>Ex.</u> $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+2^n}$ $\longrightarrow$ $\displaystyle\sum \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{\frac{1}{1+2^n}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{1+2^n} = 1$$

$\sum \frac{1}{2^n}$ conv. by Geom. Series Test, $r = \frac{1}{2}$

$\sum \frac{1}{2^n}$ conv. by Limit Comp. Test

$\therefore \sum \frac{1}{1+2^n}$ conv. by

<u>Thm.</u> Direct Comparison Test

Let $0 < a_n \leq b_n$ after some value of n.

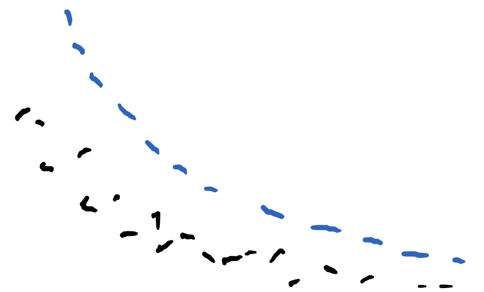
i) If $\sum b_n$ converges, then $\sum a_n$ converges.

ii) If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Informally:

1. If the "larger" series converges, then the "smaller" series must also converge.
2. If the "smaller" series diverges, then the "larger" series must also diverge.

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2} \longrightarrow \sum \frac{1}{n^2}$



$$\frac{|\cos n|}{n^2} \overset{?}{\leq} \frac{1}{n^2}$$

$$|\cos n| \overset{?}{\leq} 1 \qquad \text{yes}$$

$\sum \frac{1}{n^2}$ conv. by p-series test, $p=2$

$\therefore \sum \frac{|\cos n|}{n^2}$ conv. by Direct Comp. Test

Pract. Determine the convergence, and state the test used

1. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ Div., Integral Test

2. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{2^n - 1}$ Conv., Limit Comp. and Geom. Series Tests

3. $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ Div., Limit Comp. and p-Series Tests