Power Series

A <u>power series</u> is an infinite degree polynomial.

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

We can generalize by centering the power series at $x = c$.

$$\sum_{n=0}^{\infty} a_n (x - c)^n = a_0 + a_1(x - c) + a_2(x - c)^2 + \cdots$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$$

$$\sum_{n=0}^{\infty} (-1)^n (x+1)^n = 1 - (x+1) + (x+1)^2 - (x+1)^3 + \cdots$$

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n} = (x-2) + \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 + \cdots$$

These are functions of x.

There may be no simpler way to express these functions.

A power series is convergent at a value of x if the infinite sum converges to a finite number when evaluated at x.

The interval of convergence is the interval of x values that make the series converge. The radius of convergence is the distance away from $x = c$ that we can go to get convergence.

Special Cases

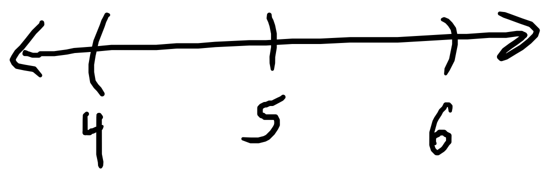
Radius $= \infty \rightarrow$ Interval $=$ all reals

Radius $= 0 \rightarrow$ Interval $=$ just the point c

Ex. Find the radius of convergence for $\sum_{n=0}^{\infty} 3(x-5)^n$

$$\lim_{n\to\infty}\left|\frac{\cancel{3}(x-5)^{n+1}}{\cancel{3}(x-5)^n}\right| = |x-5| < 1$$

$r = 1$



Ex. Find the radius of convergence for $\sum_{n=0}^{\infty} n!\,x^n$

$$\lim_{n\to\infty}\left|\frac{\cancel{(n+1)!}^{\,n+1}\ \cancel{x^{n+1}}^{\,x}}{\cancel{n!}\ \cancel{x^n}}\right| = \lim_{n\to\infty}\left|(n+1)x\right| = \infty \not< 1$$

$R = 0$

$$\lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 < 1 \quad \text{for any } x$$

$R = \infty$

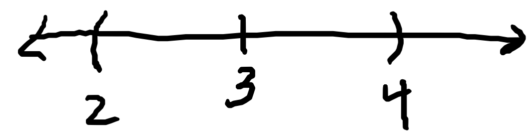
Ex. Find the interval of convergence and radius of convergence for

$$\sum_{n=1}^{\infty}\frac{(x-3)^n}{n}$$

$$\lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{n+1}\cdot\frac{n}{(x-3)^n}\right| = \lim_{n\to\infty}\left|(x-3)\underbrace{\frac{n}{n+1}}_{\to 1}\right| = |x-3| < 1$$

$\underline{x=2}$: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ conv. by alt. harm. series

$\underline{x=4}$: $\displaystyle\sum_{n=1}^{\infty}\frac{1^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$ div. by harm. series



Rad. = 1

Int.: $[2, 4)$

Ex. Find the interval of convergence and radius of convergence for

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{(-1)(x+1)}{2} \right)^n \qquad r = \frac{-(x+1)}{2} \qquad \left| \frac{-(x+1)}{2} \right| < 1 \qquad |x+1| < 2$$

$$\lim_{n\to\infty} \left| \frac{\cancel{(x+1)^{n+1}}^{\,x+1}}{\cancel{2^{n+1}}_{\,2}} \cdot \frac{\cancel{2^n}}{\cancel{(x+1)^n}} \right| = \left| \frac{x+1}{2} \right| < 1 \longrightarrow |x+1| < 2$$

$$r = 2$$

$-3 \qquad -1 \qquad 1$

int: $(-3, 1)$

$x=-3$: $\sum \frac{(-1)^n(-2)^n}{2^n} = \sum \left(\frac{(-1)(-2)}{2} \right)^n = \sum 1$

div. by Test for Div.

$x=1$: $\sum \frac{(-1)^n(2)^n}{2^n} = \sum (-1)^n$

Does this series look like something we know?

T-shirt design ideas are due next class.