# Volume

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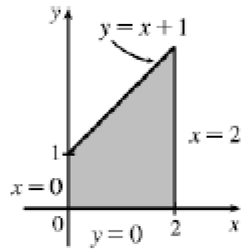
9. A cross-section is a disk with radius $x+1$, so its area is

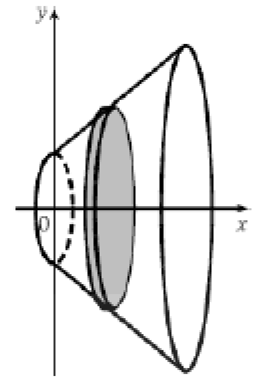
$$A(x) = \pi\left(x+1\right)^2 = \pi(x^2+2x+1).$$

$$= \pi\left(x+1\right)^2 = \pi(x^2+2x+1).$$

$$V = \int_0^2 A(x)\,dx = \int_0^2 \pi(x^2+2x+1)\,dx$$

$$= \pi\left[\tfrac{1}{3}x^3 + x^2 + x\right]_0^2 = \pi\left(\tfrac{8}{3}+4+2\right) = \tfrac{26}{3}\pi$$



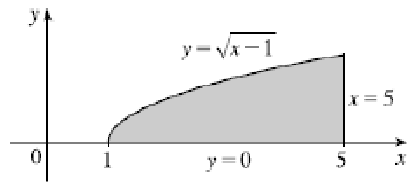

11. A cross-section is a disk with radius $\sqrt{x-1}$, so its area is

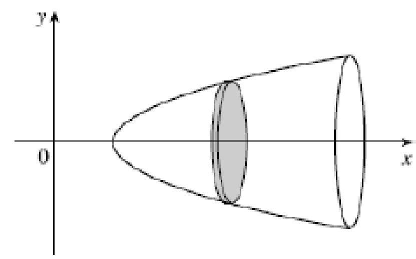
$$A(x) = \pi\left(\sqrt{x-1}\right)^2 = \pi(x-1).$$

$$V = \int_1^5 A(x)\,dx = \int_1^5 \pi(x-1)\,dx$$

$$= \pi\left[\tfrac{1}{2}x^2 - x\right]_1^5$$

$$= \pi\left[\left(\tfrac{25}{2}-5\right) - \left(\tfrac{1}{2}-1\right)\right] = 8\pi$$



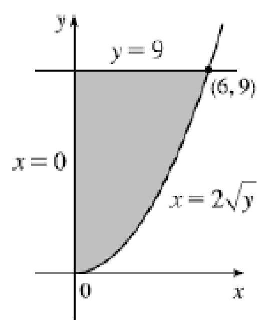



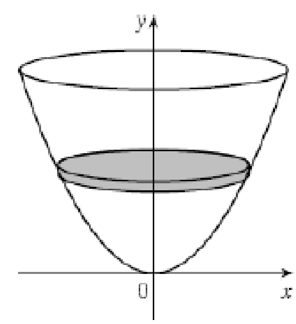

13. A cross-section is a disk with radius $2\sqrt{y}$ so its area is

$$A(y) = \pi\left(2\sqrt{y}\right)^2.$$

$$V = \int_0^9 A(y)\,dy = \pi\int_0^9 \left(2\sqrt{y}\right)^2 dy = 4\pi\int_0^9 y\,dy$$

$$= 4\pi\tfrac{1}{2}y^2\Big]_0^9 = 2\pi(81) = 162\pi$$



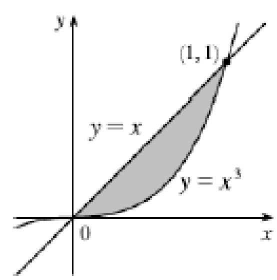



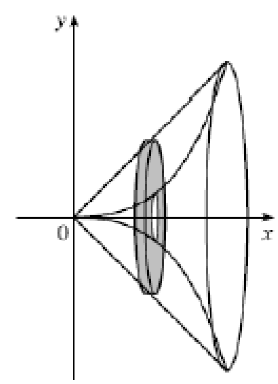

15. A cross-section is a washer (annulus) with inner radius $x^3$ and outer radius $x$ so its area is

$$A(x) = \pi(x)^2 - \pi(x^3)^2 = \pi(x^2 - x^6).$$

$$V = \int_0^1 A(x)\,dx = \pi\int_0^1 (x^2 - x^6)\,dx$$

$$= \pi\left[\tfrac{1}{3}x^3 - \tfrac{1}{7}y^7\right]_0^1 = \pi\left(\tfrac{1}{3}-\tfrac{1}{7}\right) = \tfrac{4}{21}\pi$$



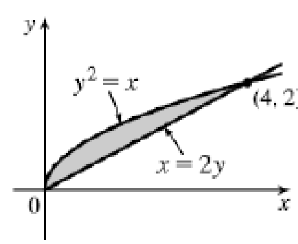



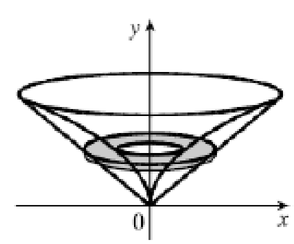

17. A cross-section is a washer with inner radius $y^2$ and outer radius $2y$ so its area is

$$A(y) = \pi(2y)^2 - \pi(y^2)^2 = \pi(4y^2 - y^4).$$

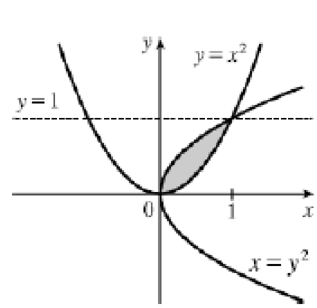
$$V = \int_0^2 A(y)\,dy = \pi\int_0^2 (4y^2 - y^4)\,dy$$

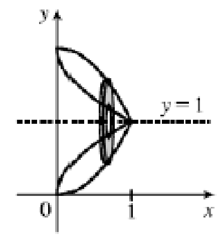
$$= \pi\left[\tfrac{4}{3}y^3 - \tfrac{1}{5}y^5\right]_0^2 = \pi\left(\tfrac{32}{3} - \tfrac{32}{5}\right) = \tfrac{64}{15}\pi$$



19. A cross-section is a washer with inner radius $1-\sqrt{x}$ and outer radius $1-x^2$ so its area is

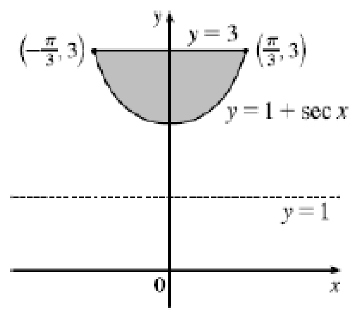


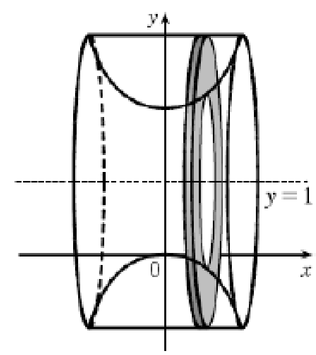




$$A(x)=\pi\left[(1-x^2)^2-\left(1-\sqrt{x}\right)^2\right]$$
$$=\pi\left[(1-2x^2+x^4)-\left(1-2\sqrt{x}+x\right)\right]$$
$$=\pi\left(x^4-2x^2+2\sqrt{x}-x\right).$$

$$V=\int_0^1 A(y)\,dy=\pi\int_0^1\left(x^4-2x^2+2x^{1/2}-x\right)dx=\pi\left[\tfrac{1}{5}x^5-\tfrac{2}{3}x^3+\tfrac{4}{3}x^{3/2}-\tfrac{1}{2}x^2\right]_0^1=\pi\left(\tfrac{1}{5}-\tfrac{2}{3}+\tfrac{4}{3}-\tfrac{1}{2}\right)=\tfrac{11}{30}\pi$$

21. A cross-section is a washer with inner radius $(1+\sec x)-1=\sec x$ and outer radius $3-1=2$, so its area is $A(x)=\pi\left[2^2-\sec^2 x\right]=\pi(4-\sec^2 x)$.



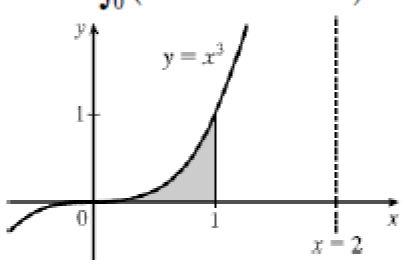



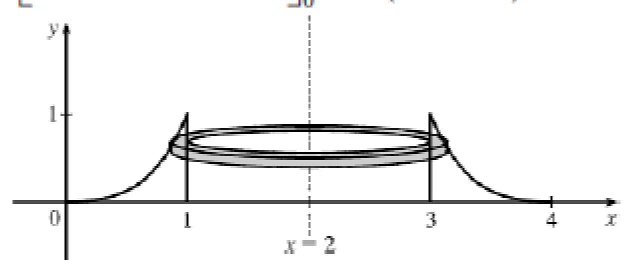

$$V=\int_{-\pi/3}^{\pi/3}A(x)\,dx=\pi\int_{-\pi/3}^{\pi/3}\left(4-\sec^2 x\right)dx$$
$$=2\pi\int_0^{\pi/3}\left(4-\sec^2 x\right)dx \quad \text{[by symmetry]}$$
$$=2\pi\left[4x-\tan x\right]_0^{\pi/3}=2\pi\left[\left(\tfrac{4\pi}{3}-\sqrt{3}\right)-0\right]$$
$$=2\pi\left(\tfrac{4\pi}{3}-\sqrt{3}\right)$$

23. A cross-section is a washer with inner radius $2-1$ and outer radius $2-\sqrt[3]{y}$, so its area is

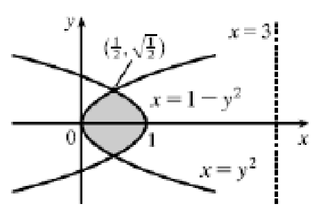
$$A(y)=\pi\left[\left(2-\sqrt[3]{y}\right)^2-(2-1)^2\right]=\pi\left[4-4\sqrt[3]{y}+\sqrt[3]{y^2}-1\right].$$

$$V=\int_0^1 A(y)\,dy=\pi\int_0^1\left(3-4y^{1/3}+y^{2/3}\right)dy \quad =\pi\left[3y-3y^{4/3}+\tfrac{3}{5}y^{5/3}\right]_0^1=\pi\left(3-3+\tfrac{3}{5}\right)=\tfrac{3}{5}\pi$$



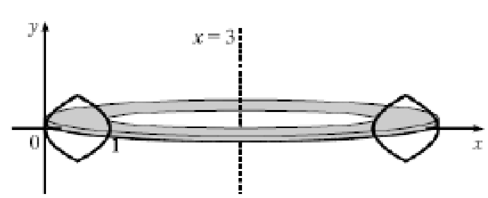




25. From the symmetry of the curves, we see that they intersect at $x=\frac{1}{2}$ and so $y^2=\frac{1}{2}\Leftrightarrow y=\pm\sqrt{\frac{1}{2}}$. A cross-section is a washer with inner radius $3-(1-y^2)$ and outer radius $3-y^2$, so its area is

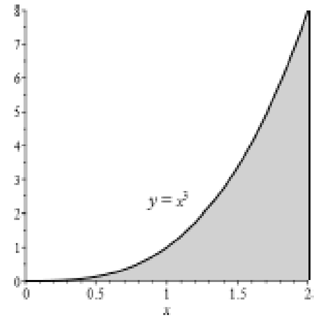


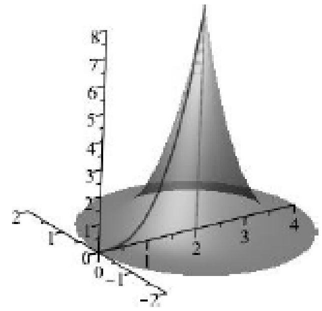

$$A(y)=\pi\left[\left(3-y^2\right)^2-\left(2+y^2\right)^2\right]$$
$$=\pi\left[(9-6y^2+y^4)-(4+4y^2+y^4)\right]=\pi(5-10y^2).$$

$$V=\int_{-\sqrt{1/2}}^{\sqrt{1/2}}A(y)\,dy=2\int_0^{\sqrt{1/2}}5\pi(1-2y^2)\,dy \quad \text{[by symmetry]}$$
$$=10\pi\left[y-\tfrac{2}{3}y^3\right]_0^{\sqrt{2}/2}=10\pi\left(\tfrac{\sqrt{2}}{2}-\tfrac{\sqrt{2}}{6}\right)=10\pi\left(\tfrac{\sqrt{2}}{3}\right)=\tfrac{10}{3}\sqrt{2}\pi$$



39. The area of this region is $A(y)=\pi\left(2-y^{1/3}\right)^2$, so the volume is $V=\int_0^8 \pi\left(2-y^{1/3}\right)^2\,dy$, option (**B**).



41. A cross-section is a washer with inner radius $x-1$ and outer radius $2\ln$, so the area is $A(y)=\pi\left[(2\ln x)^2-(x-1)^2\right]=\pi\left(4(\ln x)^2-x^2+2x-1\right)$. The curves intersect at $x=1$ and $x=a\approx 3.512862417$. Therefore,

$$V=\pi\int_1^a A(x)\,dx=\pi\int_1^a\left(4(\ln x)^2-x^2+2x-1\right)dx \overset{\text{CAS}}{\approx} 5.298,$$ which is option (**A**).

42. A cross-section is a washer with inner radius $y=\sqrt{x}\Rightarrow x=y^2$ and outer radius $y=6-x\Rightarrow x=6-y$, so its area is $A(y)=\pi(6-y)^2-\pi(y^2)^2$. The volume of the resulting solid is (**A**) $\pi\int_0^2\left[(6-y)^2-(y^2)^2\right]dy$.

43. A cross-section is a washer with radius $\sqrt{x}$ so the area is $A(x)=\pi\left(\sqrt{x}\right)^2=\pi x$, and the volume is $V=\pi\int_0^4 A(x)\,dx=\pi\int_0^4 x\,dx=\pi\tfrac{1}{2}x^2\Big]_0^4=\pi\tfrac{1}{2}(16)-0=8\pi$, which is option (**D**).

44. A cross-section is a washer with radius $\sec x$ so the area is $A(x)=\pi\sec^2 x$ and the volume is $V=\pi\int_0^{\pi/4} A(x)\,dx=\pi\int_0^{\pi/4}\sec^2 x\,dx=\pi\left[\tan x\right]_0^{\pi/4}=\pi\left[\tan\tfrac{\pi}{4}-\tan 0\right]=\pi(1-0)=\pi$, option (**A**).