p. 567: 5-25 odd, 36

5. Let $x = 2\tan\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 2\sec^2\theta\, d\theta$ and

$\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = \sqrt{4(\tan^2\theta + 1)} = \sqrt{4\sec^2\theta} = 2|\sec\theta| = 2\sec\theta$ for the relevant values of θ.

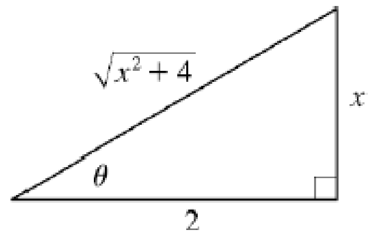
$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx = \int \frac{8\tan^3\theta}{2\sec\theta} 2\sec^2\theta\, d\theta = 8\int \tan^2\theta \sec\theta \tan\theta\, d\theta$$

$$= 8\int (\sec^2\theta - 1)\sec\theta\tan\theta\, d\theta = 8\int (u^2-1)\,du \quad [u = \sec\theta]$$

$$= 8\left(\tfrac{1}{3}u^3 - u\right) + C$$

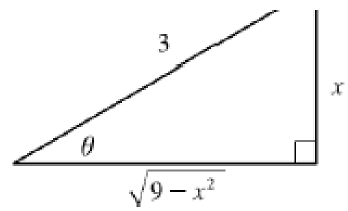
$$= \tfrac{8}{3}\sec^3\theta - 8\sec\theta + C = \tfrac{8}{3}\left(\tfrac{\sqrt{x^2+4}}{2}\right)^3 - 8\left(\tfrac{\sqrt{x^2+4}}{2}\right) + C$$

$$= \tfrac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + C \quad \text{or} \quad \tfrac{1}{3}\sqrt{x^2+4}\cdot\left(x^2-8\right) + C$$



7. Let $x = 3\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 3\cos\theta\, d\theta$ and

$\sqrt{9-x^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3|\cos\theta| = 3\cos\theta$.



$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\sin^2\theta}{3\cos\theta} 3\cos\theta\, d\theta = 9\int \sin^2\theta\, d\theta$$

$$= 9\int \tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{9}{2}\left(\theta - \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{9}{2}\theta - \tfrac{9}{4}(2\sin\theta\cos\theta) + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C$$

9. Let $u = 36 - x^2$, so $du = -2x\,dx$. When $x = 0$, $u = 36$; when $x = 3$, $u = 27$. Thus,

$$\int_0^3 \frac{x}{\sqrt{36-x^2}}\,dx = \int_{36}^{27} \frac{1}{\sqrt{u}}\left(-\frac{1}{2}du\right) = -\frac{1}{2}\left[2\sqrt{x}\right]_{36}^{27} = -\left(\sqrt{27} - \sqrt{36}\right) = 6 - 3\sqrt{3}$$

Another method: Let $x = 6\sin\theta$, so $dx = 6\cos\theta\, d\theta$, $x = 0 \Rightarrow \theta = 0$, and $x = 3 \Rightarrow \theta = \frac{\pi}{6}$. Then

$$\int_0^3 \frac{x}{\sqrt{36-x^2}}\,dx = \int_0^{\pi/6} \frac{6\sin\theta}{\sqrt{36(1-\sin^2\theta)}} 6\cos\theta\, d\theta = \int_0^{\pi/6} \frac{6\sin\theta}{6\cos\theta} 6\cos\theta\, d\theta = \int_0^{\pi/6} \sin\theta\, d\theta$$

$$= 6\left[-\cos\theta\right]_0^{\pi/6} = 6\left(-\tfrac{\sqrt{3}}{2} + 1\right) = 6 - 3\sqrt{3}$$

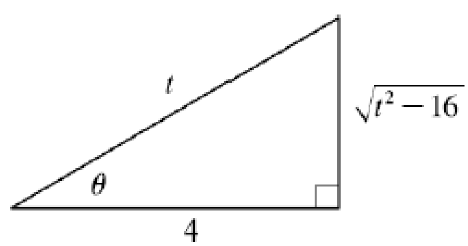
11. Let $t = 4\sec\theta$, where $0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dt = 4\sec\theta\tan\theta\, d\theta$ and

$\sqrt{t^2-16} = \sqrt{16\sec^2\theta - 16} = \sqrt{16\tan^2\theta} = 4\tan\theta$ for the relevant values of θ, so

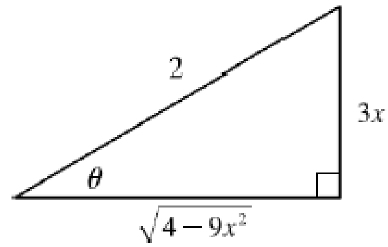
$$\int \frac{dt}{t^2\sqrt{t^2-16}} = \int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\cdot 4\tan\theta} = \frac{1}{16}\int \frac{1}{\sec\theta}\,d\theta$$

$$= \frac{1}{16}\int \cos\theta\, d\theta$$

$$= \frac{1}{16}\sin\theta + C = \frac{1}{16}\frac{\sqrt{t^2-16}}{t} + C = \frac{\sqrt{t^2-16}}{16t} + C$$



13. Let $x = \frac{2}{3}\sin\theta$, so $dx = \frac{2}{3}\cos\theta\,d\theta$, $x = 0 \Rightarrow \theta = 0$, and $x = \frac{2}{3} \Rightarrow \theta = \frac{\pi}{2}$. Thus,



$$\int_0^{2/3}\sqrt{4-9x^2}\,dx = \int_0^{\pi/2}\sqrt{4-9\cdot\tfrac{4}{9}\sin^2\theta}\,\tfrac{2}{3}\cos\theta\,d\theta$$
$$= \int_0^{\pi/2} 2\cos\theta\cdot\tfrac{2}{3}\cos\theta\,d\theta = \tfrac{4}{3}\int_0^{\pi/2}\cos^2\theta\,d\theta$$
$$= \tfrac{4}{3}\int_0^{\pi/2}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta = \tfrac{2}{3}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \tfrac{2}{3}\left[\left(\tfrac{\pi}{2}+0\right)-(0+0)\right] = \tfrac{\pi}{3}$$

15. Let $t = 2\tan\theta$, so $dt = 2\sec^2\theta\,d\theta$, $t = 0 \Rightarrow \theta = 0$, and $t = 2 \Rightarrow \theta = \frac{\pi}{4}$. Thus,

$$\int_0^2 \frac{dt}{\sqrt{4+t^2}} = \int_0^{\pi/4}\frac{2\sec^2\theta\,d\theta}{\sqrt{4+4\tan^2\theta}} = \int_0^{\pi/4}\frac{2\sec^2\theta\,d\theta}{2\sec\theta} = \int_0^{\pi/4}\sec\theta\,d\theta = \Big[\ln\left|\sec\theta+\tan\theta\right|\Big]_0^{\pi/4}$$
$$= \ln\left|\sqrt{2}+1\right| - \ln\left|1+0\right| = \ln(\sqrt{2}+1)$$

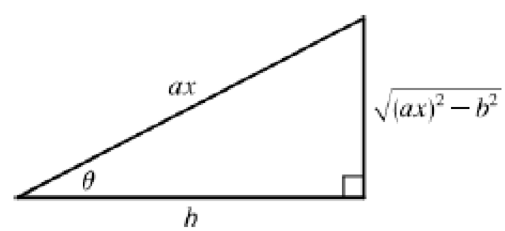
17. Let $x = \tan\theta$, so $dx = \sec^2\theta\,d\theta$, $x = 0 \Rightarrow \theta = 0$, and $x = 1 \Rightarrow \theta = \frac{\pi}{4}$. Thus,

$$\int_0^1 \frac{dx}{(x^2+1)^2} = \int_0^{\pi/4}\frac{\sec^2\theta\,d\theta}{(\tan^2\theta+1)^2} = \int_0^{\pi/4}\frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^2} = \int_0^{\pi/4}\cos^2\theta\,d\theta = \int_0^{\pi/4}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$
$$= \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \tfrac{1}{2}\left[\left(\tfrac{\pi}{4}+\tfrac{1}{2}\right)-0\right] = \tfrac{\pi}{8}+\tfrac{1}{4}$$

19. Let $x = \frac{1}{3}\sec\theta$, so $dx = \frac{1}{3}\sec\theta\tan\theta\,d\theta$, $x = \sqrt{2}/3 \Rightarrow \theta = \frac{\pi}{4}$, and $x = \frac{2}{3} \Rightarrow \theta = \frac{\pi}{3}$. Then

$$\int_{\sqrt{2}/3}^{2/3}\frac{dx}{x^5\sqrt{9x^2-1}} = \int_{\pi/4}^{\pi/3}\frac{\frac{1}{3}\sec\theta\tan\theta\,d\theta}{\left(\frac{1}{3}\right)^5\sec^5\theta\tan\theta} = 3^4\int_{\pi/4}^{\pi/3}\cos^4\theta\,d\theta = 81\int_{\pi/4}^{\pi/3}\left[\tfrac{1}{2}(1+\cos 2\theta)\right]^2 d\theta$$
$$= \tfrac{81}{4}\int_{\pi/4}^{\pi/3}(1+2\cos 2\theta+\cos^2 2\theta)\,d\theta = \tfrac{81}{4}\int_{\pi/4}^{\pi/3}\left[1+2\cos 2\theta+\tfrac{1}{2}(1+\cos 4\theta)\right]d\theta$$
$$= \tfrac{81}{4}\int_{\pi/4}^{\pi/3}\left(\tfrac{3}{2}+2\cos 2\theta+\tfrac{1}{2}\cos 4\theta\right)d\theta = \tfrac{81}{4}\left[\tfrac{3}{2}\theta+\sin 2\theta+\tfrac{1}{8}\sin 4\theta\right]_{\pi/4}^{\pi/3}$$
$$= \tfrac{81}{4}\left[\left(\tfrac{\pi}{2}+\tfrac{\sqrt{3}}{2}-\tfrac{\sqrt{3}}{16}\right)-\left(\tfrac{3\pi}{8}+1+0\right)\right] = \tfrac{81}{4}\left(\tfrac{\pi}{8}+\tfrac{7}{16}\sqrt{3}-1\right) = \tfrac{81}{64}\left(2\pi+7\sqrt{3}-16\right)$$

21. Let $ax = b\sec\theta$, so $(ax)^2 = b^2\sec^2\theta \Rightarrow$
$(ax)^2 - b^2 = b^2\sec^2\theta - b^2 = b^2(\sec^2\theta - 1) = b^2\tan^2\theta$.
So $\sqrt{(ax)^2-b^2} = b\tan\theta$, $dx = \dfrac{b}{a}\sec\theta\tan\theta\,d\theta$, and

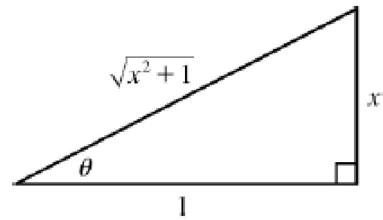


$$\int\frac{dx}{\left[(ax)^2-b^2\right]^{3/2}} = \int\frac{\frac{b}{a}\sec\theta\tan\theta}{b^3\tan^3\theta}\,d\theta = \frac{1}{ab^2}\int\frac{\sec\theta}{\tan^2\theta}\,d\theta$$
$$= \frac{1}{ab^2}\int\frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{ab^2}\int\csc\theta\cot\theta\,d\theta = -\frac{1}{ab^2}\csc\theta + C$$
$$= -\frac{1}{ab^2}\frac{ax}{\sqrt{(ax)^2-b^2}} + C = -\frac{x}{b^2\sqrt{(ax)^2-b^2}} + C$$

23. $u = 1 + x^2$, so $du = 2x\,dx$. Then

$$\int \frac{x}{\sqrt{1+x^2}}\,dx = \int \frac{1}{\sqrt{u}}\left(\frac{1}{2}du\right) = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\cdot 2u^{1/2} + C = \sqrt{1+x^2} + C$$

25. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$ and $\sqrt{x^2+1} = \sec\theta$, and $x = 0 \Rightarrow \theta = 0$, $x = 1 \Rightarrow \theta = \frac{\pi}{4}$, so



$$\int_0^1 \sqrt{x^2+1}\,dx = \int_0^{\pi/4} \sec\theta\sec^2\theta\,d\theta = \int_0^{\pi/4}\sec^3\theta\,d\theta$$

$$= \tfrac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4}$$

$$= \tfrac{1}{2}\Big[\sqrt{2}\cdot 1 + \ln(1+\sqrt{2}) - 0 - \ln(1+0)\Big] = \tfrac{1}{2}\Big[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\Big]$$

36. The average value of $f(x) = \sqrt{x^2-1}/x$ on the interval $[1,7]$ is

$$\frac{1}{7-1}\int_1^7 \frac{\sqrt{x^2-1}}{x}\,dx = \frac{1}{6}\int_0^\alpha \frac{\tan\theta}{\sec\theta}\cdot\sec\theta\tan\theta\,d\theta \qquad \left[\begin{array}{l}\text{where } x = \sec\theta,\, dx = \sec\theta\tan\theta\,d\theta,\\ \sqrt{x^2-1} = \tan\theta, \text{ and } \alpha = \sec^{-1}7\end{array}\right]$$

$$= \tfrac{1}{6}\int_0^\alpha \tan^2\theta\,d\theta = \tfrac{1}{6}\int_0^\alpha\left(\sec^2\theta - 1\right)d\theta = \tfrac{1}{6}\big[\tan\theta - \theta\big]_0^\alpha$$

$$= \tfrac{1}{6}(\tan\alpha - \alpha) = \tfrac{1}{6}\left(\sqrt{48} - \sec^{-1}7\right)$$