<u>7.4 and 7.5</u>

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13. $\int \frac{3t-2}{t+1}dt = \int \left(3 - \frac{5}{t+1}\right)dt = 3t - 5\ln|t+1| + C$

15. $\frac{y}{(y+4)(2y-1)} = \frac{A}{y+4} + \frac{B}{2y-1}$. Multiply both sides by $(y+4)(2y-1)$ to get $y = A(2y-1) + B(y+4) \Rightarrow\ y = 2Ay - A + By + 4B \Rightarrow y = (2A+B)y + (-A+4B)$. The coefficients of y must be equal and the constant terms are also equal, so $2A+B=1$ and $-A+4B=0$. Adding these equations gives us $9B=1 \Leftrightarrow B=\frac{1}{9}$, and hence, $A=\frac{4}{9}$. Thus,

$$\int \frac{y\,dy}{(y+4)(2y-1)} = \int \left(\frac{\frac{4}{9}}{y+4} + \frac{\frac{1}{9}}{2y-1}\right)dy = \tfrac{4}{9}\ln|y+4| + \tfrac{1}{9}\cdot\tfrac{1}{2}\ln|2y-1| + C$$
$$= \tfrac{4}{9}\ln|y+4| + \tfrac{1}{18}\ln|2y-1| + C.$$

Another method: Substituting $\frac{1}{2}$ for y in the equation $y = A(2y-1) + B(y+4)$ gives $\frac{1}{2} = \frac{9}{2}B \Leftrightarrow B = \frac{1}{9}$. Substituting -4 for y gives $-4 = -9A \Leftrightarrow A = \frac{4}{9}$.

17. $\frac{x-4}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}$. Multiply both sides by $(x-2)(x-3)$ to get $x-4 = A(x-3) + B(x-2) \Rightarrow$ $x-4 = Ax - 3A + Bx - 2B \Rightarrow x-4 = (A+B)x + (-3A-2B)$.

The coefficients of x must be equal and the constant terms are also equal, so $A+B=1$ and $-3A-2B=-4$. Adding twice the first equation to the second gives us $-A=-2 \Leftrightarrow A=2$, and hence, $B=-1$. Thus,

$$\int_0^1 \frac{x-4}{x^2-5x+6}dx = \int_0^1 \left(\frac{2}{x-2} - \frac{1}{x-3}\right)dx = \Big[2\ln|x-2| - \ln|x-3|\Big]_0^1$$
$$= (0 - \ln 2) - (2\ln 2 - \ln 3) = -3\ln 2 + \ln 3\ \left[\text{or } \ln\tfrac{3}{8}\right].$$

19. If $a \neq b$, $\frac{1}{(x+a)(x+b} = \frac{1}{b-a}\left(\frac{1}{x+a} - \frac{1}{x+b}\right)$, so if $a \neq b$, then

$$\int \frac{dx}{(x+a)(x+b)} = \frac{1}{b-a}\left(\ln|x+a| - \ln|x+b|\right) + C = \frac{1}{b-a}\ln\left|\frac{x+a}{x+b}\right| + C.$$

If $a=b$, then $\int \frac{dx}{(x+a)^2} = -\frac{1}{x+a} + C$.

21. $\frac{x^3+4x^2+x-1}{x^3+x^2} = 1 + \frac{3x^2+x-1}{x^2(x+1)}$. Write $\frac{3x^2+x-1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$. Multiplying both sides by $x^2(x+1)$ gives $3x^2 + x - Ax(x+1) + B(x+1) + Cx^2$. Substituting 0 for x gives $-1 = B$. Substituting -1 for x gives $1 = C$. Equating coefficients of x^2 gives $3 = A + C = A + 1$, so $A = 2$. Thus,

$$\int_1^2 \frac{x^3+4x^2+x-1}{x^3+x^2}dx = \int_1^2 \left(\frac{2}{x} - \frac{1}{x^2} + \frac{1}{x+1}\right)dx = \left[x + 2\ln|x| + \frac{1}{x} + \ln|x+1|\right]_1^2$$
$$= (2 + 2\ln 2 + \tfrac{1}{2} + \ln 3) - (1 + 0 + 1 + \ln 2) = \tfrac{1}{2} + \ln 2 + \ln 3,\ \text{or}\ \tfrac{1}{2} + \ln 6.$$

23. $\dfrac{3x^2+6x+2}{x^2+3x+2}=3+\dfrac{-3x-4}{(x+1)(x+2)}$. Write $\dfrac{-3x-4}{(x+1)(x+2)}=\dfrac{A}{x+1}+\dfrac{B}{x+2}$. Multiplying both sides by $(x+1)(x+2)$ gives $-3x-4=A(x+2)+B(x+1)$. Substituting –2 for x gives $2=-B \Leftrightarrow B=-2$. Substituting –1 for x gives $-1=A$. Thus,

$$\int_1^2 \frac{3x^2+6x+2}{x^2+3x+2}\,dx=\int_1^2\left(3+\frac{1}{x+1}-\frac{2}{x+2}\right)dx=\left[3x-\ln|x+1|-2\ln|x+2|\right]_1^2$$
$$=(6-\ln 3-2\ln 4)-(3-\ln 2-2\ln 3)=3+\ln 2+\ln 3-2\ln 4=3+\ln\tfrac{3}{8}$$

25. $\dfrac{x(3-5x)}{(3x-1)(x-1)^2}=\dfrac{A}{3x-1}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$. Multiplying both sides by $(3x-1)(x-1)^2$ gives $x(3-5x)=A(x-1)^2+B(x-1)(3x-1)+C(3x-1)^2$. Substituting 1 for x gives $-2=2C \Leftrightarrow C=-1$. Substituting $\frac{1}{3}$ for x gives $\frac{4}{9}=\frac{4}{9}A \Leftrightarrow A=1$. Substituting 0 for x gives $0=A+B-C=1+B+1$,. so $B=-2$. Thus,

$$\int_2^3 \frac{x(3-5x)}{(3x-1)(x-1)^2}\,dx=\int_2^3\left(\frac{1}{3x-1}-\frac{2}{x-1}-\frac{1}{(x-1)^2}\right)dx=\left[\tfrac{1}{3}\ln|3x-1|-2\ln|x-1|+\frac{1}{x-1}\right]_2^3$$
$$=\left(\tfrac{1}{3}\ln 8-2\ln 2+\tfrac{1}{2}\right)-\left(\tfrac{1}{3}\ln 5-0+1\right)=-\ln 2-\tfrac{1}{3}\ln 5-\tfrac{1}{2}$$

27. $$\int\frac{x^4+9x^2+x+2}{x^2+9}\,dx=\int\left(x^2+\frac{x+2}{x^2+9}\right)dx=\int\left(x^2+\frac{x}{x^2+9}+\frac{2}{x^2+9}\right)dx$$
$$=\tfrac{1}{3}x^3+\tfrac{1}{2}\ln\left(x^2+9\right)+\tfrac{2}{3}\tan^{-1}\tfrac{x}{3}+C$$

29. $\dfrac{x^2-x+6}{x^3+3x}=\dfrac{x^2-x+6}{x(x^2+3)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+3}$. Multiply both sides by $x(x^2+3)$ to get $x^2-x+6=A(x^2+3)+(Bx+C)x$. Substituting 0 for x gives $6=3A \Leftrightarrow A=2$. The coefficients of the x^2-terms must be equal, so $1=A+B \Rightarrow B=1-2=-1$. The coefficients of the x-terms must be equal, so $-1=C$. Thus,

$$\int\frac{x^2-x+6}{x^3+3x}\,dx=\int\left(\frac{2}{x}+\frac{-x-1}{x^2+3}\right)dx=\int\left(\frac{2}{x}-\frac{x}{x^2+3}-\frac{1}{x^2+3}\right)dx$$
$$=2\ln|x|-\tfrac{1}{2}\ln(x^2+3)-\tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{x}{\sqrt{3}}\right)+C.$$

31. $$\int\frac{x^2+x+1}{(x^2+1)^2}\,dx=\int\frac{x^2+1}{(x^2+1)^2}\,dx+\int\frac{x}{(x^2+1)^2}\,dx=\int\frac{1}{x^2+1}\,dx+\tfrac{1}{2}\int\frac{1}{u^2}\,du \quad \left[u=x^2+1,\ du=2x\,dx\right]$$
$$=\tan^{-1}x+\tfrac{1}{2}\left(-\frac{1}{u}\right)+C=\tan^{-1}x-\frac{1}{2(x^2+1)}+C.$$

33. $\dfrac{x^3+6x-2}{x^4+6x^2}=\dfrac{x^3+6x-2}{x^2(x^2+6)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{Cx+D}{x^2+6}$. Multiply both sides by $x^2(x^2+6)$ to get

$x^3+6x-2=Ax(x^2+6)+B(x^2+6)+(Cx+D)x^2 \Leftrightarrow$

$x^3+6x-2=Ax^3+6Ax+Bx^2+6B+Cx^3+Dx^2 \Leftrightarrow x^3+6x-2=(A+C)x^3+(B+D)x^2+6Ax+6B.$

Substituting 0 for x gives $-2=6B \Leftrightarrow B=-\frac{1}{3}$. Equating coefficients of x^2 gives $0=B+D$, so $D=\frac{1}{3}$.

Equating coefficients of x gives $6=6A \Leftrightarrow A=1$. Equating coefficients of x^3 gives $1=A+C$,

so $C=0$. Thus,

$$\int\frac{x^3+6x-2}{x^4+6x^2}dx=\int\left(\frac{1}{x}=\frac{-\frac{1}{3}}{x^2}+\frac{\frac{1}{3}}{x^2+6}\right)dx=\ln|x|+\frac{1}{3x}+\frac{1}{3\sqrt{6}}\tan^{-1}\left(\frac{x}{\sqrt{6}}\right)+C.$$

44. $\displaystyle\int\frac{dx}{x^2-9}=\int\frac{dx}{6(x-3)}-\int\frac{dx}{6(x+3)}=\frac{1}{6}\ln|x-3|-\frac{1}{6}\ln|x+3|+C=\frac{1}{6}\ln\frac{|x-3|}{|x+3|}+C$, option (**C**).

45. $\displaystyle\int\frac{dx}{x^2+2x+2}=\int\frac{dx}{(x+1)^2}=\tan^{-1}(x+1)+C$, choice (**B**).

46. $\displaystyle\int_0^4\frac{x-1}{x^2-4x-5}dx=\int_0^4\frac{dx}{3(x+1)}+\int_0^4\frac{2dx}{3(x-5)}=\frac{1}{3}\ln|x+1|+\frac{2}{3}\ln|x-5|\Big|_0^4=\left(\frac{\ln 5}{3}-\frac{2\ln 5}{3}\right)=-\frac{\ln 5}{3}$, (**C**).

47. $\displaystyle\int_0^1\frac{10x+18}{x^2+3x+2}dx=\int_0^1\frac{2dx}{(x+2)}+\int_0^1\frac{8dx}{(x+1)}=\ln\left((x+2)^2\cdot(x+1)^8\right)\Big|_0^1=\ln\left(\frac{9\cdot 256}{4\cdot 1}\right)=\ln 576$, choice (**A**).

48. $\displaystyle\int_0^1\frac{dx}{x^2+6x+5}=\int_0^1\frac{dx}{4(x+1)}-\int_0^1\frac{dx}{4(x+5)}=\frac{1}{4}\ln\left(\frac{|x+1|}{|x+5|}\right)\Bigg|_0^1=\frac{\ln\left(\frac{1}{3}\right)-\ln\left(\frac{1}{5}\right)}{4}=\frac{\ln\left(\frac{5}{3}\right)}{4}$, choice (**C**).

63. Let $u=\tan^{-1}x, dv=x\,dx \Rightarrow du=dx/(1+x^2), v=\frac{1}{2}x^2$.

Then $\displaystyle\int x\tan^{-1}x\,dx=\tfrac{1}{2}x^2\tan^{-1}x-\frac{1}{2}\int\frac{x^2}{1+x^2}dx$. To evaluate the last integral, use long division or

observe that $\displaystyle\int\frac{x^2}{1+x^2}dx=\int\frac{(1+x^2)-1}{1+x^2}dx=\int 1\,dx-\int\frac{1}{1+x^2}dx=x-\tan^{-1}x+C_1$. So

$\displaystyle\int x\tan^{-1}x\,dx=\tfrac{1}{2}x^2\tan^{-1}x-\tfrac{1}{2}\left(x-\tan^{-1}x+C_1\right)=\tfrac{1}{2}\left(x^2\tan^{-1}x+\tan^{-1}x-x\right)+C.$

76. $\dfrac{3}{x^2-x-2}=\dfrac{1}{x-2}+\dfrac{-1}{x-1}$, so $A=1$ and $B=-1$ and $A+B=0$, option (**B**).

77. $\dfrac{3}{(x^2+2)(x-1)}=\dfrac{1}{x-1}+\dfrac{-x-1}{x^2+2}$, so $A=1$ and $B=C=-1$ and $A+B+C=-1$, option (**B**).

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9. Let $u=3x+1$. Then $du=3\,dx \Rightarrow$

$\displaystyle\int_0^1(3x+1)^{\sqrt{2}}dx=\tfrac{1}{3}\int_1^4 u^{\sqrt{2}}du=\tfrac{1}{3}\left[\tfrac{1}{\sqrt{2}+1}u^{\sqrt{2}+1}\right]_1^4=\tfrac{1}{3\sqrt{2}+1}\left(4^{\sqrt{2}+1}-1\right)$

13. Let $u = 2x+1$. Then $du = 2\,dx \Rightarrow$

$$\int_0^1 \frac{x}{(2x+1)^3}\,dx = \int_1^3 \frac{(u-1)/2}{u^3}\left(\tfrac{1}{2}\,dx\right)u = \frac{1}{4}\int_1^3 \left(\frac{1}{u^2} - \frac{1}{u^3}\right)du = \frac{1}{4}\left[-\frac{1}{u} + \frac{1}{2u^2}\right]_1^3$$

$$= \tfrac{1}{4}\left[\left(-\tfrac{1}{3} + \tfrac{1}{18}\right) - \left(-1 + \tfrac{1}{2}\right)\right] = \tfrac{1}{4}\left(\tfrac{2}{9}\right) = \tfrac{1}{18}$$

17. Let $u = \dfrac{1}{x}, dv = \dfrac{\cos(1/x)}{x^2}dx \Rightarrow du = -\dfrac{1}{x^2}dx,\ v = -\sin\left(\dfrac{1}{x}\right)$. Then

$$\int \frac{\cos(1/x)}{x^3}\,dx = -\frac{1}{x}\sin\left(\frac{1}{x}\right) - \int \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx = -\frac{1}{x}\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) + C.$$

21. Let $u = \ln(1+x^2), dv = dx \Rightarrow du = \dfrac{2x}{1+x^2}dx, v = x$. Then

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}dx = x\ln(1+x^2) - 2\int \frac{(x^2+1)-1}{1+x^2}dx$$

$$= x\ln(1+x^2) - 2\int\left(1 - \frac{1}{1+x^2}\right)dx = x\ln(1+x^2) - 2x + 2\tan^{-1}x + C$$

25. Let $u = \sqrt{t}$. Then $du = \dfrac{1}{2\sqrt{t}}dt \Rightarrow \displaystyle\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}dt = \int_1^2 e^u(2\,du) = 2\left[e^u\right]_1^2 = 2\left(e^2 - e\right)$.

29. Let $u = +(\ln x)^2$ so that $du = \dfrac{2\ln x}{x}dx$. Then

$$\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\left(2\sqrt{u}\right) + C = \sqrt{1+(\ln x)^2} + C.$$

33. $$\int_0^1 \frac{3x^2+1}{x^3+x^2+x+1}dx = \int_0^1\left(\frac{2}{x+1} + \frac{x-1}{x^2+1}\right)dx = \int_0^1\left(\frac{2}{x+1} + \frac{x}{x^2+1} - \frac{1}{x^2+1}\right)dx$$

$$= \left[2\ln|x+1| + \tfrac{1}{2}\ln\left(x^2+1\right) - \tan^{-1}x\right]_0^1 = \left(2\ln 2 + \tfrac{1}{2}\ln 2 - \tfrac{\pi}{4}\right) - (0+0-0)$$

$$= \tfrac{5}{2}\ln 2 - \tfrac{\pi}{4}$$

37. $$\left|e^x - 1\right| = \begin{cases} e^x - 1 & \text{if } e^x - 1 \ge 0 \\ -\left(e^x - 1\right) & \text{if } e^x - 1 < 0 \end{cases} = \begin{cases} e^x - 1 & \text{if } x \ge 0 \\ 1 - e^x & \text{if } x < 0 \end{cases}$$

Thus, $\displaystyle\int_{-1}^2 \left|e^x - 1\right|dx = \int_{-1}^0\left(1 - e^x\right)dx + \int_0^2\left(e^x - 1\right)dx = (x - e^x)\Big|_{-1}^0 + \left(e^x - x\right)\Big|_0^2$

$$= (0-1) - \left(-1 - e^{-1}\right) + \left(e^2 - 2\right) - (1-0) = e^2 + e^{-1} - 3$$

41. $$\int_{\pi/4}^{\pi/2} \frac{1+4\cot x}{4-\cot x}dx = \int_{\pi/4}^{\pi/2} \frac{(1+4\cos x/\sin x)}{(4-\cos x/\sin x)}\cdot\frac{\sin x}{\sin x}dx = \int_{\pi/4}^{\pi/2}\frac{\sin x + 4\cos x}{4\sin x - \cos x}dx$$

$$= \int_{3\sqrt{2}}^4 \frac{du}{u} \qquad \begin{bmatrix} u = 4\sin x - \cos x, \\ du = (4\cos x + \sin x)dx \end{bmatrix}$$

$$= \left[\ln|u|\right]_{3\sqrt{2}}^4 = \ln 4 - \ln\frac{3}{\sqrt{2}} = \ln\frac{4}{3/\sqrt{2}} = \ln\left(\frac{4}{3}\sqrt{2}\right)$$

45. $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sin\theta\cot\theta}{\sec\theta}\,d\theta = \int_{\pi/6}^{\pi/3} \cos^2\theta\,d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/3}(1+\cos 2\theta)\,d\theta = \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Bigg|_{\pi/6}^{\pi/3}$

$$= \frac{1}{2}\left[\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right)-\left(\frac{\pi}{6}+\frac{\sqrt{3}}{4}\right)\right] = \frac{1}{2}\left(\frac{\pi}{6}\right) = \frac{\pi}{12}$$

49. Let $u = \tan^{-1} x,\ dv = \dfrac{1}{x^2}dx \Rightarrow du = \dfrac{1}{1+x^2}dx,\ v = -\dfrac{1}{x}$. Then

$$I = \int \frac{\tan^{-1} x}{x^2}dx = -\frac{1}{x}\tan^{-1} x - \int\left(-\frac{1}{x(1+x^2)}\right)dx = -\frac{1}{x}\tan^{-1} x + \int\left(\frac{A}{x}+\frac{Bx+C}{1+x^2}\right)dx$$

So $C = 0$, $A = 1$, and $A + B = 0 \Rightarrow B = -1$. Thus,

$$I = -\frac{1}{x}\tan^{-1} x + \int\left(\frac{1}{x}-\frac{x}{1+x^2}\right)dx = -\frac{1}{x}\tan^{-1} x + \ln|x| - \frac{1}{2}\ln\left|1+x^2\right| + C$$

$$= -\frac{\tan^{-1} x}{x} + \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$$

53. Use integration by parts with $u = (x-1)e^x,\ dv = \dfrac{1}{x^2}dx \Rightarrow du = \left[(x-1)e^x + e^x\right]dx = xe^x dx,\ v = -\dfrac{1}{x}$.

Then $\displaystyle\int \frac{(x-1)e^x}{x^2}dx = (x-1)e^x\left(-\frac{1}{x}\right) - \int -e^x\,dx = -e^x + \frac{e^x}{x} + e^x + C = \frac{e^x}{x} + C.$