9.3 and 9.4

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11. The function $f(x)=x^{-0.3}$ is continuous, positive and decreasing on $[1,\infty)$, so the Integral Test applies.

$\int_1^\infty x^{-0.3}\,dx=\lim_{b\to\infty}\int_b^\infty x^{-0.3}\,dx=\lim_{b\to\infty}\left[\frac{x^{0.7}}{0.7}\right]_1^b=\lim_{b\to\infty}\left(-\frac{b^{0.7}}{0.7}-\frac{1}{0.7}\right)=\infty.$ Because this improper integral diverges, the series $\sum_{n=1}^{\infty}n^{-0.3}$ also diverges by the Integral Test.

15. The function $f(x)=x^2e^{-x^3}$ is continuous, positive and decreasing (*) on $[1,\infty)$, so the Integral Test applies. $\int_1^\infty x^2e^{-x^3}\,dx=\lim_{b\to\infty}\int_b^\infty x^2e^{-x^3}\,dx=\lim_{b\to\infty}\left[-\frac{1}{3}e^{-x^3}\right]_1^b=-\frac{1}{3}\cdot\lim_{b\to\infty}\left(e^{-b^3}-e^{-1}\right)=-\frac{1}{3}\left(0-e^{-1}\right)=\frac{1}{3e}.$

Because this improper integral is convergent, the series $\sum_{n=1}^{\infty}n^2e^{-n^3}$ is also convergent.

$$(*)\quad f'(x)=x^2e^{-x^3}(-3x^2)+e^{-x^3}(2x)=xe^{-x^3}(-3x+2)=\frac{x(2-3x^3)}{e^{x^3}}<0 \text{ for } x>1$$

19. $\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots=\sum_{n=1}^{\infty}\frac{1}{2n+3}$. The function $f(x)=\frac{1}{2x+3}$ is continuous, positive and decreasing on $[1,\infty)$, so the Integral Test applies.

$$\int_1^\infty\frac{1}{2x+3}dx=\lim_{b\to\infty}\int_b^\infty\frac{1}{2x+3}dx=\lim_{b\to\infty}\left[\tfrac{1}{2}\ln(2x+3)\right]_1^b=\lim_{b\to\infty}\tfrac{1}{2}\cdot\left(\ln(2b+3)-\ln 5\right)=\infty.$$

Because this improper integral is divergent, the series $\sum_{n=1}^{\infty}\frac{1}{2n+3}$ is also divergent by the Integral Test.

23. The function $f(x)=\frac{\sqrt{x}}{1+x^{3/2}}$ is continuous, and positive on $[1,\infty)$, so the Integral Test applies.

$f'(x)=\frac{(1+x^{3/2})\left(\frac{1}{2}x^{-1/2}\right)-x^{-1/2}\left(\frac{3}{2}x^{1/2}\right)}{(1+x^{3/2})^2}=\frac{\frac{1}{2}x^{-1/2}+\frac{1}{2}x-\frac{3}{2}x}{(1+x^{3/2})^2}=\frac{1-2x^{3/2}}{2\sqrt{x}(1+x^{3/2})^2}<0$ for $x\ge 1$, so f is decreasing on $[1,\infty)$, and the Integral Test applies.

$\int_1^\infty\frac{\sqrt{x}}{1+x^{3/2}}dx=\lim_{b\to\infty}\int_b^\infty\frac{\sqrt{x}}{1+x^{3/2}}dx=\lim_{b\to\infty}\left[\tfrac{2}{3}\ln(1+x^{3/2})\right]_1^b=\lim_{b\to\infty}\left[\tfrac{2}{3}\ln(1+b^{3/2})-\tfrac{2}{3}\ln 2\right]=\infty,$ so the series $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{1+n^{3/2}}$ diverges.

27. The function $f(x)=\frac{3x-4}{x^2-2x}=\frac{2}{x}+\frac{1}{x-2}$ [by partial fractions] is continuous, positive and decreasing on $[3,\infty)$ because it is the sum of two such functions, so we can apply the Integral Test.

$$\int_3^\infty\frac{3x-4}{x^2-2x}dx=\lim_{b\to\infty}\int_3^\infty\left[\frac{2}{x}+\frac{1}{x-2}\right]dx=\lim_{b\to\infty}\left[2\ln x+\ln(x-2)\right]_3^b=\lim_{b\to\infty}\left[2\ln b+\ln(b-2)-2\ln 3\right]=\infty.$$

The integral is divergent, so the series $\sum_{n=1}^{\infty}\frac{3n-4}{n^2-2n}$ is divergent.

31. The function $f(x) = xe^{-x^2} = \frac{x}{e^{x^2}}$ is continuous and positive on $[1,\infty)$, and also decreasing since

$f'(x) = \frac{e^{x^2} \cdot - xe^{x^2} \cdot 2x}{(e^{x^2})^2} = \frac{1-2x^2}{e^{x^2}} < 0$ for $x > \sqrt{\frac{1}{2}} \approx 0.7$, so we can use the Integral Test on $[1,\infty)$.

$\int_1^\infty xe^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b xe^{-x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_1^b = \lim_{b\to\infty}\left[-\frac{1}{2}e^{-b^2}+\frac{1}{2}e^{-1}\right] = \frac{1}{2e}$, so the series $\sum_{k=1}^{\infty} ke^{-k^2}$ converges.

40. (a) If $f(x) = \frac{10x}{e^x}$, then $f'(x) = \frac{e^x(10) - 10x\cdot e^x}{(e^x)^2} = \frac{10e^x(1-x)}{(e^x)^2} = \frac{10(1-x)}{e^x} < 0$ for $x > 1$. Thus, f is ultimately decreasing.

(b) $\int_1^\infty f(x)\,dx = \lim_{b\to\infty}\int_1^b 10xe^{-x}\,dx = \lim_{b\to\infty}\left(\left[-10xe^{-x}\right]_1^b - \int_1^b -10e^{-x}\,dx\right) = \lim_{b\to\infty}\left(\frac{-10b}{e^b} + \frac{10}{e} - \left[10e^{-x}\right]_1^b\right)$

$$= \lim_{b\to\infty}\left(\frac{-10b}{e^b} + \frac{10}{e} - \frac{10}{e^b} + \frac{10}{e}\right) \overset{H}{=} \left(0 + \frac{20}{e} - 0\right) = \frac{20}{e}.$$

(c) The function $f(x) = \frac{10x}{e^x}$ is continuous, positive and decreasing on $[1,\infty)$. so we can apply the Integral Test. Because $\int_1^\infty f(x)\,dx$ converges, the series $\sum_{n=1}^{\infty}\frac{10n}{e^n}$ must also converge.

41. The series $\sum_{n=3}^{\infty}\frac{1}{n(\ln n)^p}$ converges for $p > 1$, option (**A**). (See Exercise 36.)

42. $\lim_{n\to\infty}\frac{0.05n}{20n+3} = \lim_{n\to\infty}\frac{0.05}{20+3/n} = 0.0025 > 0$, so series I diverges by the nth term test. Similarly, $\lim_{n\to\infty}\left(\frac{e}{\sin(2)}\right)^n = \infty$, so series II also diverges by the same test. The function $f_3(x) = \frac{x}{x^2+1}$ is continuous, positive and increasing on $[1,\infty)$, so we may apply the Integral Test.

$\int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{b\to\infty}\int_1^b \frac{x}{x^2+1}\,dx = \lim_{b\to\infty}\left[\frac{1}{2}\ln(x^2+1)\right]_1^b = \frac{1}{2}\left(\ln(b^2+1) - \ln 2\right) = \infty$, so series III diverges.
Therefore, option (**D**) is correct.

43. Series (A) converges by Exercise 29. Series (B) is a constant multiple of the convergent p-series $\sum_{n=1}^{\infty}\frac{10}{n^2}$ with $p = 2 > 1$, so it converges. Series (D) can be shown to converge by the Integral Test. The function $f(x) = \frac{1}{x^{\ln 2}}$ is continuous, positive and decreasing on $[1,\infty)$, so we can apply the Integral Test. $\int_1^\infty \frac{1}{x^{\ln 2}}\,dx = \lim_{b\to\infty}\int_1^b x^{-\ln 2}\,dx = \lim_{b\to\infty}\left[\frac{1}{1-\ln 2}x^{1-\ln 2}\right]_1^b = \lim_{b\to\infty}\frac{1}{1-\ln 2}\left(b^{1-\ln 2} - 1\right) = \infty$, so series (**C**) $\sum_{n=1}^{\infty}\frac{1}{n^{\ln 2}}$ also diverges.

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11. $\frac{1}{\sqrt{n}-1}>\frac{1}{\sqrt{n}}$ for all $n\geq 2$, so $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}-1}$ diverges by comparison with $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, which diverges because it is a p-series with $p=\frac{1}{2}\leq 1$.

15. $\frac{6^n}{5^n-1}>\frac{6^n}{5^n}=\left(\frac{6}{5}\right)^n$ for all $n\geq 1$. $\sum_{n=1}^{\infty}\left(\frac{6}{5}\right)^n$ is a convergent geometric series $\left(|r|=\frac{6}{5}>1\right)$, so $\sum_{n=1}^{\infty}\frac{6^n}{5^n-1}$ diverges by the Comparison Test.

19. $\frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}<\frac{2k(k^2)}{k(k^2)^2}<\frac{2k^3}{k^5}=\frac{2}{k^2}$ for all $k\geq 1$, so $\sum_{n=1}^{\infty}\frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$ converges by comparison with $2\sum_{n=1}^{\infty}\frac{1}{k^2}$, which converges because it is a constant multiple of a p-series with $p=2>1$.

23. $\frac{1}{n^n}\leq\frac{1}{n^2}$ for all $n\geq 1$, so $\sum_{n=1}^{\infty}\frac{1}{n^n}$ converges by comparison with $\sum_{n=1}^{\infty}\frac{1}{n^2}$, which converges because it is a p-series with $p=2>1$.

27. Use the Limit Comparison Test with $a_n=\frac{n^2+n+1}{n^4+n^2}$ and $b_n=\frac{1}{n^2}$:

$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{(n^2+n+1)n^2}{n^2(n^2+1)}=\lim_{n\to\infty}\frac{n^2+n+1}{n^2+1}=\lim_{n\to\infty}\frac{1+1/n+1/n^2}{1+1/n^2}=1>0.$ Because $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series $[p=2>1]$, the series $\sum_{n=1}^{\infty}\frac{n^2+n+1}{n^4+n^2}$ also converges.

31. $\frac{n+3^n}{n+2^n}>\frac{3^n}{n+2^n}>\frac{3^n}{2^n+2^n}=\frac{3^n}{2\cdot 2^n}=\frac{1}{2}\left(\frac{3}{2}\right)^n$ for all $n\geq 1$, so the series $\sum_{n=1}^{\infty}\frac{n+3^n}{n+2^n}$ diverges by comparison with $\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$, which is a constant multiple of a divergent geometric series $\left(|r|=\frac{3}{2}>1\right)$.

Or: Use the Limit Comparison Test with $a_n=\frac{n+3^n}{n+2^n}$ and $b_n=\left(\frac{3}{2}\right)^n$.

44. If $\{a_n\}$ and $\{b_n\}$ are sequences of positive constants with $a_n>b_n$ for all $n=1,2,3,\cdots$, then if $\sum_{n=1}^{\infty}a_n$ converges, $\sum_{n=1}^{\infty}b_n$ must also converge, which means $\lim_{n\to\infty}b_n=0$. This is choice (**D**).

45. $\frac{8\sqrt{n}-2}{2n^2}<\frac{8\sqrt{n}}{2n^2}\leq 4\frac{\sqrt{n}}{n^2}\leq 4\frac{1}{n^{3/2}}$, so $\sum_{n=1}^{\infty}\frac{8\sqrt{n}-2}{2n^2}$ converges by comparison with the series $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$, which is a p-series with $\left[p=\frac{3}{2}>1\right]$. This is choice (**B**).

46. $\frac{6n+5}{(n-3)^p}\leq\frac{6n}{(n-3)^p}\leq\frac{6}{(n-3)^{p-1}}$, which is a p-series, so it converges for $p-1>1\Leftrightarrow p>2$, choice (**D**).

47. $n^2-5n+7\geq n^2$ for all $n\geq 1$, so $\frac{1}{n^2-5n+7}\leq\frac{1}{n^2-5n+7}$ for all $n\geq 1$. The series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges because it is a p-series $[p=2>1]$, so $\sum_{n=1}^{\infty}\frac{1}{n^2-5n+7}$ also converges. This is series (**C**).

48. $\frac{(0.95)^n + 7(0.6)^n}{n} = \frac{(0.95)^n}{n} + 7 \cdot \frac{(0.6)^n}{n}$. $\lim_{n\to\infty} \left| \frac{(0.95)^{n+1}}{n+1} \cdot \frac{n}{(0.95)^n} \right| = \lim_{n\to\infty} \left((0.95)^n \cdot \frac{n}{n+1} \right)$

$= \lim_{n\to\infty} (0.95)^n \cdot \frac{1}{1 + \frac{1}{n}} = 0 < 1$, so $\sum_{n=1}^{\infty} \frac{(0.95)^n}{n}$ converges by the Ratio Test (see Section 9.6). Similarly,

$\lim_{n\to\infty} \left| \frac{7(0.6)^{n+1}}{n+1} \cdot \frac{n}{7(0.6)^n} \right| = \lim_{n\to\infty} \left((0.6)^n \cdot \frac{n}{n+1} \right) = 0 < 1$, so $\sum_{n=1}^{\infty} \frac{7(0.6)^n}{n}$ also converges by the Ratio Test.

Therefore, $\sum_{n=1}^{\infty} \frac{(0.95) + 7(0.6)^n}{n}$ converges. This is option (**D**).

49. If $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges then $\sum_{n=1}^{\infty} \frac{k}{a_n}$ converges by the Limit Comparison Test using $A_n = \frac{k}{a_n}$ and $B_n = \frac{1}{a_n}$

because $\lim_{n\to\infty} \frac{A_n}{B_n} = \lim_{n\to\infty} \frac{k / a_n}{1 / a_n} = \lim_{n\to\infty} \frac{ka_n}{a_n} = k > 0$. If we let $A_n = \frac{k}{a_n}$ and $B_n = \frac{1}{k \cdot a_n}$, then

$\lim_{n\to\infty} \frac{A_n}{B_n} = \lim_{n\to\infty} \frac{1 / k \cdot a_n}{1 / a_n} = \lim_{n\to\infty} \frac{a_n}{k \cdot a_n} = \frac{1}{k} > 0$, so $\sum_{n=1}^{\infty} \frac{1}{k \cdot a_n}$ also converges by the Limit Comparison Test.

Finally, suppose $a_n = n^2$. Then $\sum_{n=1}^{\infty} \frac{1}{a_n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, but $\sum_{n=1}^{\infty} \frac{n}{k \cdot n^2} = \sum_{n=1}^{\infty} \frac{1}{k \cdot n}$ diverges because it is a constant multiple of the divergent harmonic series. Thus, the correct choice is (**B**), I and II.

50. Given that $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges, if $a_n = \frac{1}{3^n + n}$ and $b_n = \frac{1}{3^n}$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{3^n + n} \cdot \frac{3^n}{1} = \lim_{n\to\infty} \frac{3^n}{3^n + n}$

$= \lim_{n\to\infty} \frac{1}{1 + \frac{n}{3^n}} = 1 > 0$, so converges $\sum_{n=1}^{\infty} \frac{1}{3^n + n}$ by the Limit Comparison Test. In addition, if we let,

$a_n = \frac{1}{4^n}$ and $b_n = \frac{1}{3^n}$, then $\lim_{n\to\infty} \frac{1}{4^n} \cdot \frac{3^n}{1} = \lim_{n\to\infty} \left(\frac{3}{4}\right)^n = 0$. By Exercise 54 (a), because $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges, so

does $\sum_{n=1}^{\infty} \frac{1}{4^n}$. However, we cannot use the Limit Comparison Test to determine the convergence of

$\sum_{n=1}^{\infty} \frac{n^n}{3^n}$ (which diverges). Thus, the correct choice is (**C**).