<u>9.6 and 9.7</u>

p. 706: 17-29 odd, 35-47 odd, 48-51, 55-59

17. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(-2)^n}\right| = \lim_{n\to\infty}\left|(-2)\cdot\frac{n^2}{(n+1)^2}\right| = 2\cdot\lim_{n\to\infty}\frac{1}{(1+1/n)^2} = 2 > 1$, so the series $\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}$ is divergent by the Ratio Test.

19. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}}{[2(n+1)+1]!}\cdot\frac{(2n+1)!}{(-3)^n}\right| = \lim_{n\to\infty}\left|(-3)\frac{1}{(2n+3)(2n+2)}\right| = 3\cdot\lim_{n\to\infty}\frac{1}{(2n+3)(2n+2)}$ $= 3(0) = 0 < 1$, so the series $\sum_{n=1}^{\infty}\frac{(-3)^n}{(2n+1)!}$ is absolutely convergent by the Ratio Test.

21. $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)e^{-(k+1)}}{ke^{-k}}\right| = \lim_{k\to\infty}\left(\frac{k+1}{k}\cdot e^{-1}\right) = \frac{1}{e}\lim_{k\to\infty}\frac{1+1/k}{1} = \frac{1}{e}(1) = \frac{1}{e} < 1$, so the series $\sum_{k=1}^{\infty}ke^{-k}$ is absolutely convergent by the Ratio Test. Because the terms of this series are positive, absolute convergence is the same as convergence.

23. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{100^{n+1}}\cdot\frac{100^n}{n!}\right| = \lim_{n\to\infty}\frac{n+1}{100} = \infty$, so the series $\sum_{n=1}^{\infty}\frac{n!}{100^n}$ diverges by the Ratio Test.

25. $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\left|\frac{n^{10}}{(-1)^{n+1}}\right| = \lim_{n\to\infty} n^{10} = \infty \neq 0$, so the series $\sum_{n=1}^{\infty}\frac{n^{10}}{(-1)^{n+1}}$ diverges by the Test for Divergence.

27. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right| = \lim_{n\to\infty}\frac{(n+1)n^n}{(n+1)^{n+1}} = \lim_{n\to\infty}\frac{n^n}{n} = \lim_{n\to\infty}\frac{1}{(1+1/n)^n} = \frac{1}{e} < 1$, so the series $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ is absolutely convergent by the Ratio Test.

29. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[2(n+1)]!}{[(n+1)!]^2}\cdot\frac{(n!)^2}{(2n)!}\right| = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \lim_{n\to\infty}\frac{(2+2/n)(2+1/n)}{(1+1/n)(1+1/n)} = \frac{2\cdot 2}{1\cdot 1} = 4 > 1$, so the series $\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$ diverges by the Ratio Test.

35. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{(-2)^n}{n^n}\right|} = \lim_{n\to\infty}\frac{2}{n} = 0 < 1$, so the series $\sum_{n=1}^{\infty}\frac{(-2)^n}{n^n}$ is absolutely convergent by the Root Test.

37. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{-2n}{n+1}\right)^{5n}\right|} = \lim_{n\to\infty}\frac{2^5n^5}{(n+1)^5} = 32\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^5 = 32\cdot\lim_{n\to\infty}(1+1/n)^5 = 32\cdot(1) = 32 > 1$, so the series $\sum_{n=0}^{\infty}\left(\frac{-2n}{n+1}\right)^{5n}$ diverges by the Root Test.

39. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{|(\arctan n)^n|} = \lim_{n\to\infty}\arctan n = \frac{\pi}{2} > 1$, so the series $\sum_{n=0}^{\infty}(\arctan n)^n$ diverges by the Root Test.

41. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{1-n}{2+3n}\right)^n\right|} = \lim_{n\to\infty}\frac{n-1}{3n+2} = \lim_{n\to\infty}\frac{1-1/n}{3+2/n} = \frac{1}{3} < 1$, so the series $\sum_{n=1}^{\infty}\left(\frac{1-n}{2+3n}\right)^n$ is absolutely convergent by the Root Test.

43. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\cdot 5^{2n+2}}{10^{n+2}}\cdot\frac{10^{n+1}}{n\cdot 5^{2n}}\right| = \lim_{n\to\infty}\frac{5^2(n+1)}{10n} = \frac{5}{2}\lim_{n\to\infty}(1+1/n) = \frac{5}{2}(1) = \frac{5}{2} > 1$, so the series $\sum_{n=1}^{\infty}\frac{n5^{2n}}{10^{n+1}}$ diverges by the Ratio Test. *Or*: Because $\lim_{n\to\infty} a_n = \infty$, the series diverges by the Test for Divergence.

45. $\left|\frac{\sin(n\pi)/6}{1+n\sqrt{n}}\right| \le \frac{1}{1+n\sqrt{n}} < \frac{1}{n^{3/2}}$, so the series $\sum_{n=1}^{\infty}\frac{\sin(n\pi)/6}{1+n\sqrt{n}}$ converges by comparison with the convergent p-series $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ $(p=\frac{3}{2}>1)$. It follows that the given series is absolutely convergent.

47. The function $f(x)=\frac{1}{x\ln x}$ is continuous, positive and decreasing on $[2,\infty)$.
$\int_2^{\infty}\frac{1}{x\ln x}dx = \lim_{b\to\infty}\int_2^b\frac{1}{x\ln x}dx = \lim_{b\to\infty}[\ln(\ln x)]_2^b = \lim_{b\to\infty}[\ln(\ln b)-\ln(\ln 2)] = \infty$, so the series $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$ diverges by the Integral Test. Now $\{b_n\}=\left\{\frac{1}{n\ln n}\right\}$ with $n\ge 2$ is a decreasing sequence of positive terms and $\lim_{n\to\infty} b_n = 0$. Thus, $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$ converges by the Alternating Series Test. It follows that $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$ is conditionally convergent.

48. If $a_n=\frac{n^2}{2^n}$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2}\right| = \lim_{n\to\infty}\left(\frac{1}{2}\frac{n^2+2n+1}{n^2}\right) = \frac{1}{2}\cdot\lim_{n\to\infty}\left(1+2/n+1/n^2\right) = \frac{1}{2}$, (**B**).

49. $\{b_n\}=\left\{\frac{n}{n^2+1}\right\}$ is a decreasing sequence of positive terms and $\lim_{n\to\infty} b_n = 0$, so $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}n}{n^2+1}$ converges by the Alternating Series Test. The function $f(x)=\frac{x}{x^2+1}$ is continuous, positive and decreasing on $[1,\infty)$. $\int_1^{\infty}\frac{x}{x^2+1}dx = \lim_{b\to\infty}\int_2^b\frac{x}{x^2+1}dx = \lim_{b\to\infty}\left[\frac{1}{2}\ln\left(x^2+1\right)\right]_2^b = \lim_{b\to\infty}[\ln(b^2+1)-\ln(2)] = \infty$, so the series $\sum_{n=2}^{\infty}\frac{n}{n^2+1}$ diverges by the Integral Test. Thus series (I), $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}n}{n^2+1}$, converges conditionally.
$\lim_{n\to\infty}\frac{n^n}{n!}\ne 0$, and $\lim_{n\to\infty}\frac{\sqrt{n^3}}{n+1}\ne 0$, so series (II) and (III) diverge by the Test for Divergence. Therefore the correct choice is (**A**), I only.

50. $\lim_{n\to\infty} \sqrt[n]{\left(\frac{1+n}{n}\right)^n} = \lim_{n\to\infty} \left(\frac{1+n}{n}\right) = \lim_{n\to\infty} (1+1/n) = 1$, so the Root Test is inconclusive for the series $\sum_{n=1}^{\infty}\left(\frac{1+n}{n}\right)^n$. $\lim_{n\to\infty} \sqrt[n]{\left(\frac{2+n}{n}\right)^n} = \lim_{n\to\infty}\left(\frac{2+n}{n}\right) = \lim_{n\to\infty}(2+1/n) = 2 > 1$, so by the Root Test, the series $\sum_{n=1}^{\infty}\left(\frac{2+n}{n}\right)^n$ diverges. $\lim_{n\to\infty} \sqrt[n]{\left(\frac{1+n}{2n}\right)^{2n}} = \lim_{n\to\infty}\left(\frac{1+n}{2n}\right)^2 = \lim_{n\to\infty}\left(\frac{1+1/n}{2}\right)^2 = \frac{1}{2} < 1$, so by the Root Test, the series $\sum_{n=1}^{\infty}\left(\frac{1+n}{2n}\right)^{2n}$ converges. Thus, the correct option is (C), III only.

51. By the recursive definition, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{5n+1}{4x+3}\right| = \frac{5}{4} > 1$, so the series diverges by the Ratio Test.

55. Statement (A) is not true because the harmonic series diverges but $\lim_{n\to\infty}\frac{1}{n} = 0$. Statement (B) is not true because it is missing the requirements that $f(x)$ be continuous, positive and decreasing. Statement (D) is not true because it is missing the requirement that $|a_{n+1}| < |a_n|$ for the Alternating Series Test. However, statement (C) is true as $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$ is the necessary condition for the Ratio Test.

56. Both $\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}$ and $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}}{n^2+1}$ converge by the Alternating Series Test. The function $f(x) = \frac{x}{x^2+1}$ is continuous, positive and decreasing on $[1,\infty)$.

$\int_1^{\infty}\frac{x}{x^2+1}dx = \lim_{b\to\infty}\int_2^b \frac{x}{x^2+1}dx = \lim_{b\to\infty}\left[\frac{1}{2}\ln\left(x^2+1\right)\right]_2^b = \lim_{b\to\infty}[\ln(b^2+1) - \ln(2)] = \infty$, so the series $\sum_{n=2}^{\infty}\frac{n}{n^2+1}$ diverges by the Integral Test. But by the Integral Test, $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$ converges. Thus, the correct option is (C): Series I is conditionally convergent, while series II is absolutely convergent.

57. $\{b_n\} = \left\{\frac{n+2}{3n^2+5}\right\}$ is a decreasing sequence of positive terms and $\lim_{n\to\infty} b_n = 0$. Thus, $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n+2}{3n^2+5}$ converges by the Alternating Series Test. The function $f(x) = \frac{x+2}{3x^2+5}$ is continuous, positive and decreasing on $[1,\infty)$. $\int_0^{\infty}\frac{x+2}{3x^2+5}dx = \lim_{b\to\infty}\int_0^b\frac{x+2}{3x^2+5}dx = \lim_{b\to\infty}\left[\frac{1}{6}\ln\left(3x^2+5\right) + \frac{2}{\sqrt{15}}\tan^{-1}\left(\frac{3x}{\sqrt{15}}\right)\right]_0^b$

$= \lim_{b\to\infty}\left[\frac{1}{6}\ln(3b^2+5) + \frac{2}{\sqrt{15}}\tan^{-1}\left(\frac{3b}{\sqrt{15}}\right)\right] - \left(\frac{1}{6}\ln(5)+0\right) = \infty$, so $\sum_{n=0}^{\infty}\frac{n+2}{3n^2+5}$ diverges by the Integral Test. Thus series (C), $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n+2}{3n^2+5}$, converges conditionally.

58. Observe that $|a_3| = \frac{1}{27} < |a_4| = \frac{1}{16}$, so it is not true that $|a_{n+1}| < |a_n|$ for all n, $\lim_{n\to\infty} a_n = 0$. Thus, statements (A) and (D) are not true. However, $\sum_{n=1}^{\infty} a_n = \left(\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \right)$. The first series converges by comparison with the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ and the second series converges by comparison with the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. The sum of two convergent series converges, so the series $\sum_{n=1}^{\infty} a_n$ must converge. The correct statement is (**C**).

59. (**A**) $\lim_{n\to\infty} \left| \frac{1/(n+1)^3}{1/n^3} \right| = \lim_{n\to\infty} \frac{n^3}{(n+1)^3} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^3} = 1$. Inclusive.

(B) $\lim_{n\to\infty} \left| \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \right| = \lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{2} \right) = \frac{1}{2} < 1$. Conclusive (convergent)

(C) $\lim_{n\to\infty} \left| \frac{(-3)^n}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-3)^{n-1}} \right| = 3 \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 3 \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n}}} = 3 > 1$. Conclusive (divergent)

(**D**) $\lim_{n\to\infty} \left| \frac{\sqrt{n+1}}{1+(n+1)^2} \cdot \frac{1+n^2}{\sqrt{n}} \right| = \lim_{n\to\infty} \left[\sqrt{1+\frac{1}{n}} \cdot \frac{\frac{1}{n^2}+1}{\frac{1}{n^2}+(1+\frac{1}{n})^2} \right] = 1$. Inconclusive

The Ratio Test is inconclusive for Statements (**A**) and (**D**).

p. 710: 5-33 EOO, 42-43, 45-48

5. Use the Limit Comparison Test with $a_n = \frac{n^2-1}{n^3+1}$ and $b_n = \frac{1}{n}$:

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n^2-1)n}{n^3+1} = \lim_{n\to\infty} \frac{n^3-n}{n^3+1} = \lim_{n\to\infty} \frac{1-1/n^2}{1+1/n^3} = 1 > 0$. Because $\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series, the series $\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+1}$ also diverges.

9. $\lim_{x\to\infty} e^x = \infty$, $\lim_{n\to\infty} x^2 = \infty$ and $\lim_{n\to\infty} 2x = \infty$, so by l'Hopital's Rule (twice), $\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$. Thus, the series $\sum_{n=1}^{\infty} \frac{n}{n^2}$ diverges by the Test for Divergence.

13. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\pi^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{\pi^{2n}} \right| = \lim_{n\to\infty} \frac{\pi^2}{(2n+2)(2n+1)} = 0 < 1$, so the series $\sum_{n=1}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$ absolutely convergent (and therefore convergent) by the Ratio Test.

17. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2} \right| = \lim_{n\to\infty} \frac{3(n+1)^2}{(n+1)n^2} = 3 \lim_{n\to\infty} \frac{n+1}{n^2} = 0 < 1$, so the series $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$ converges by the Ratio Test.

21. $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\left|\frac{1\cdot3\cdot5\cdot\dots\cdot(2n-1)(2n+1)}{2\cdot5\cdot8\cdot\dots\cdot(3n-1)(3n+2)}\cdot\frac{2\cdot5\cdot8\cdot\dots\cdot(3n-1)}{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}\right| = \lim_{n\to\infty}\frac{2n+1}{3n+2} = \lim_{n\to\infty}\frac{2+1/n}{3+2/n} = \frac{2}{3} < 1$, so the series $\sum_{n=1}^{\infty}\frac{2^{k-1}3^{k+1}}{k^k}$ =converges by the Ratio Test.

25. $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\left|(-1)^n\cos(1/n^2)\right| = \lim_{n\to\infty}\left|\cos(1/n^2)\right| = \cos 0 = 1$, so the series $\sum_{n=1}^{\infty}(-1)^n\cos(1/n^2)$ diverges by the Test for Divergence.

29. Use the Ratio Test. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{e^{(n+1)^2}}\cdot\frac{e^{n^2}}{n!}\right| = \lim_{n\to\infty}\left(\frac{(n+1)!e^{n^2}}{e^{n^2+2n+1}n!}\right) = \lim_{n\to\infty}\left(\frac{n+1}{e^{2n+1}}\cdot\frac{1}{5}\right) = 0 < 1$, so $\sum_{n=1}^{\infty}\frac{n!}{e^{n^2}}$ converges

33. Let $f(x) = \frac{\sqrt{x}}{x+5}$. Then $f(x)$ is continuous and positive on $[1,\infty)$ and since $f'(x) = \frac{5-x}{2\sqrt{x}(x+5)^2} < 0$ for $x > 5$, $f(x)$ is eventually decreasing, so we can use the Alternating Series Test.

$\lim_{n\to\infty}\frac{\sqrt{n}}{n+5} = \lim_{n\to\infty}\frac{1}{n^{1/2}+5n^{-1/2}} = 0$ so the series $\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+5}$ converges.

42. Because $f(x) = \frac{1}{x+2}$ is continuous, positive and decreasing on $[1,\infty)$, and $\int_1^{\infty}\frac{1}{x+2} = \infty$, we can use the Integral Test to show that the series diverges. We can also show the series diverges using the Limit Comparison Test with $a_n = \frac{1}{n}$, and $b_n = \frac{1}{n+2}$.

Then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1/n}{1/n+2} = \lim_{n\to\infty}\frac{n+2}{n} = \lim_{n\to\infty}(1+2/n) = 1 > 0$, and $\{a_n\}$ is the harmonic series so it diverges. Therefore $\{b_n\}$ also diverges. However, we cannot use the Ratio test because $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1/(n+3)}{1/(n+2)} = \lim_{n\to\infty}\frac{n+2}{n+3} = \lim_{n\to\infty}\frac{1+2/n}{1+3/n} = 1$, so the Ratio Test is inconclusive. Therefore, option (**B**) is correct.

43. $\lim_{n\to\infty}\left|\frac{(n+1)^3}{2^{n+1}}\cdot\frac{2^n}{n^3}\right|=\lim_{n\to\infty}\left|2\cdot\left(\frac{n+1}{n}\right)^3\right|=2\cdot\lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^3\right|=2\cdot\lim_{n\to\infty}\left|1+\frac{3}{n}+\frac{3}{n^2}+\frac{3}{n^3}\right|=2\cdot 1=2.$ Thus, we can test **(A)** for convergence using the Ratio Test.

However, $\lim_{n\to\infty}\left|\frac{(n+1)^3}{(n+1)!}\cdot\frac{n!}{n^3}\right|=\lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^3\cdot\frac{1}{(n+1)}\right|=\lim_{n\to\infty}\left|\frac{\left(1+\frac{1}{n}\right)^3}{(n+1)}\right|=2\cdot\lim_{n\to\infty}\left|\frac{1+\frac{3}{n}+\frac{3}{n^2}+\frac{3}{n^3}}{(n+1)}\right|=\frac{1}{0+1}=1,$

and $\lim_{n\to\infty}\left|\frac{2(n+1)+1}{(n+1)^2+3}\cdot\frac{n^2+3}{2n+1}\right|=\lim_{n\to\infty}\left|\frac{2n+3}{2n+1}\cdot\frac{n^2+3}{n^2+2n+4}\right|=\lim_{n\to\infty}\left(\frac{2n+3}{2n+1}\right)\cdot\lim_{n\to\infty}\left(\frac{n^2+3}{n^2+2n+4}\right)$

$$=\lim_{n\to\infty}\left(\frac{2+3/n}{2+1/n}\right)\cdot\lim_{n\to\infty}\left(\frac{1+3/n^2}{1+2/n+4/n^2}\right)=1\cdot 1=1, \text{ and}$$

$\lim_{n\to\infty}\left|\frac{\ln(n+1)}{(n+1)^2}\cdot\frac{n^2}{\ln(n)}\right|=\lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)^2\cdot\frac{\ln(n+1)}{\ln n}\right|=\lim_{n\to\infty}\left(\frac{1}{(1+1/n)}\right)^2\cdot\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n}=1\cdot\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n}.$

Because $\ln x$ is continuous for $x\geq 1$ and $\lim_{x\to\infty}\ln(x+1)=\lim_{n\to\infty}\ln(x)=\infty,$ we can use l'Hopital's Rule to find $\lim_{x\to\infty}\frac{\ln(x+1)}{\ln x}=\lim_{x\to\infty}\frac{1/(x+1)}{1/x}=\lim_{x\to\infty}\frac{x}{x+1}=1.\ \cdot\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n}$. Therefore $\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n}=1,$ and d

45. Only series (**D**), $\sum_{n=1}^{\infty}(-1)^n\frac{n+3}{5n+(0.1)^n},$ does not converge. By the Divergent Series Test,

$\lim_{n\to\infty}\frac{n+3}{5n+(0.1)^n}=\lim_{n\to\infty}\frac{1+3/n}{5+(0.1)^n/n}=\frac{1}{5}\neq 0.$ $\frac{(0.1)^n}{0.2n}=5\frac{(0.1)^n}{n}\leq 5(0.1)^n$ and $\sum_{n=1}^{\infty}5(0.1)^n$ converges because it is a geometric series with $r=0.1<1.$ Therefore series (A), $\sum_{n=1}^{\infty}\frac{(0.1)^n}{0.2n},$ also converges. Series (B) converges by comparison with the sum of two geometric series $\left(\sum_{n=1}^{\infty}4(0.7)^n \text{ and } \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n\right)$ because

$\frac{4(0.7)^n+(1.5)^{-n}}{n}\leq 4(0.7)^n+(1.5)^{-n}=4(0.7)^n+\left(\frac{2}{3}\right)^n.$ For series (C), as $n\to\infty,(0.2)^n\to 0,$ so we can consider only $\sum_{n=1}^{\infty}\frac{5}{3n^2}=\frac{5}{3}\sum_{n=1}^{\infty}\frac{1}{n^2}$ which converges because it is a constant multiple of the convergent p-series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ $(p=2>1).$

46. The series converges, so statement (A) is not true. The series is not a geometric series, so (B) is not true. The series does fail the hypotheses of the Alternating Series Test, but that does not imply that the Series diverges. Statement (**D**) is true: The function $f(x)=xe^{-x}$ is continuous, positive, and decreasing on $[2,\infty)$, and using parts we find

$$\int_2^{\infty}xe^{-x}\,dx=\lim_{b\to\infty}\int_2^{b}xe^{-x}\,dx=\lim_{b\to\infty}\left(-xe^{-x}+e^{-x}\right)\Big|_2^b=\lim_{b\to\infty}\left(\frac{-b}{e^b}+\frac{1}{e^b}\right)-\left(\frac{-2}{e^2}+\frac{1}{e^2}\right)=0+\frac{3}{e^b}=\frac{3}{e^b}.$$

47. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{8n} = \frac{1}{8} \cdot \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \frac{1}{8} \cdot \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{8} \cdot \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges since it is a multiple of a p-series with $p = ½ < 1$. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges (see Exercise 11). $\lim_{n\to\infty} \frac{n+2}{6n+4} = \lim_{n\to\infty} \frac{1+2/n}{6+4/n} = \frac{1}{6} \neq 0$ so $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{n+2}{6n+4}$ diverges by the Test for Divergence. However, **(D)**, $\sum_{n=1}^{\infty} \frac{\sqrt{6n}}{3n^2+4}$ converges because $\frac{\sqrt{6n}}{3n^2+4} < \frac{\sqrt{6n}}{3n^2} = \frac{\sqrt{6}}{3} \cdot \frac{\sqrt{n}}{n^2} = \frac{\sqrt{6}}{3} \cdot \frac{1}{n^{3/2}}$ and $\frac{\sqrt{6}}{3} \cdot \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges since it is a multiple of a p-series with $p = \frac{3}{2} > 1$.

48. The first series is a geometric series with $|r| = \frac{k}{5}$, and in order for it to converge we need $0 \le k < 5$. $\frac{2n^2+5}{n^k+7} \le \frac{2n^2+5}{n^k} = \frac{2}{n^{k-2}} + \frac{5}{n^k} \Rightarrow \sum_{n=1}^{\infty} \frac{5}{n^k}$ converges for $k > 1$ (it is a multiple of a p-series), and $\sum_{n=1}^{\infty} \frac{2}{n^{k-2}}$ will converge for $k - 2 > 1$ or $k > 3$. Thus both $\sum_{n=1}^{\infty} \left(\frac{k}{n}\right)^5$ and $\sum_{n=1}^{\infty} \frac{2n^2+5}{n^k+7}$ will converge only if $k = 4$, which is option **(C)**.