p. 717: 13-29 EOO, 37-40, 45-47

13. If $a_n = \dfrac{(-1)^n x^n}{\sqrt[3]{n}}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-1)^{n+1}x^{n+1}}{\sqrt[3]{n+1}}\cdot\dfrac{\sqrt[3]{n}}{(-1)^n x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-1)x\sqrt[3]{n}}{\sqrt[3]{n+1}}\right| = \lim\limits_{n\to\infty}\sqrt[3]{\dfrac{1}{1+1/n}}\,|x| = |x|$.

By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n x^n}{\sqrt[3]{n}}$ converges when $|x|<1$, so $R=1$.

When $x=1$, the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{\sqrt[3]{n}}$ converges by the Alternating Series Test. When $x=1$, the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt[3]{n}}$ diverges because it is a p-series $\left(p=\frac{1}{3}\le 1\right)$. Thus, the interval of convergence is $I=(-1,1]$.

17. If $a_n = \dfrac{x^n}{n^4 4^n}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)^4 4^{n+1}}\cdot\dfrac{n^4 4^n}{x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{n^4}{(n+1)^4}\cdot\dfrac{x}{4}\right| = \lim\limits_{n\to\infty}\left(\dfrac{n}{n+1}\right)^4\cdot\dfrac{|x|}{4}$

$=1^4\cdot\dfrac{|x|}{4}=\dfrac{|x|}{4}$. By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\dfrac{x^n}{n^4 4^n}$ converges when $\dfrac{|x|}{4}<1 \Leftrightarrow |x|<4$, so $R=4$.

When $x=4$, the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^4}$ converges since it is a p-series $(p=4>1)$. When $x=-4$, the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n^4}$ converges by the Alternating Series Test. Thus, the interval of convergence is $I=[-4,4]$.

21. If $a_n = \dfrac{n}{2^n(n^2+1)}x^n$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)x^{n+1}}{2^{n+1}(n^2+2n+2)}\cdot\dfrac{2^n(n^2+1)}{nx^n}\right| = \lim\limits_{n\to\infty}\dfrac{n^3+n^2+n+1}{n^3+2n^2+2n}\cdot\dfrac{|x|}{2}$

$=\lim\limits_{n\to\infty}\dfrac{1+1/n+1/n^2+1/n^3}{1+2/n+2/n^2}\cdot\dfrac{|x|}{2}=\dfrac{|x|}{2}$. By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\dfrac{n}{2^n(n^2+1)}x^n$ converges when $\dfrac{|x|}{2}<1 \Leftrightarrow |x|<2$, so $R=2$. When $x=2$, the series $\sum\limits_{n=1}^{\infty}\dfrac{n}{n^2+1}$ diverges by the Limit Comparison Test with $b_n=\dfrac{1}{n}$. When $x=-2$, the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n n}{n^2+1}$ converges by the Alternating Series Test. Thus, the interval of convergence is $I=[-2,2)$.

25. If $a_n = \dfrac{(x+2)^n}{2^n \ln n}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(x+2)^{n+1}}{2^{n+1}\ln(n+1)}\cdot\dfrac{2^n \ln n}{(x+2)^n}\right| = \lim\limits_{n\to\infty}\dfrac{\ln n}{\ln(n+1)}\cdot\dfrac{|x+2|}{2}=\dfrac{|x+2|}{2}$ because by l'Hopital's Rule, $\lim\limits_{n\to\infty}\dfrac{\ln n}{\ln(n+1)}=\lim\limits_{n\to\infty}\dfrac{\ln x}{\ln(x+1)}=\lim\limits_{n\to\infty}\dfrac{1/x}{1/(x+1)}=\lim\limits_{n\to\infty}\dfrac{x+1}{x}=\lim\limits_{n\to\infty}(1+1/x)=1$. By the Ratio Test, the series $\sum\limits_{n=2}^{\infty}\dfrac{(x+2)^n}{2^n\ln n}$ converges when $\dfrac{|x+2|}{2}<1 \Leftrightarrow |x+2|<2 \Leftrightarrow -2<x+2<1 \Leftrightarrow$ $-4<x<0$, and $R=2$. When $x=-4$, the series $\sum\limits_{n=2}^{\infty}\dfrac{(-1)^n}{\ln n}$ converges by the Alternating Series Test.

When $x=0$, the series $\sum\limits_{n=2}^{\infty}\dfrac{1}{\ln n}$ diverges by the Limit Comparison Test with $b_n=\dfrac{1}{n}$ (or by comparison with the harmonic series). Thus, the interval of convergence is $I=$

29. If $a_n = n!(2x-1)^n$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(2x-1)^{n+1}}{n!(2x-1)^n}\right| = \lim_{n\to\infty}|(n+1)(2x-1)| = |2x-1|\cdot\infty > 0$, so by the Ratio Test, the series $\sum_{n=1}^{\infty} a_n = n!(2x-1)^n$ converges only for $|2x-1| = 0 \Leftrightarrow 2x = 1 \Leftrightarrow x = \frac{1}{2}$.

Therefore, the radius of convergence is $R = 0$, and the interval of convergence is $I = \{\frac{1}{2}\}$.

37. The correct choice is (**D**): Series II and III do not have radii of convergence $R = \infty$. Using the Ratio Test for Series I, $\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{(n+1)!}\cdot\frac{n!}{(x-1)^n}\right| = |x-1|\lim_{n\to\infty}\left(\frac{1}{n+1}\right) = |x-1|\cdot 0 = 0 \Rightarrow R = \infty$. Applying the Ratio Test to Series II, we find $\lim_{n\to\infty}\left|\frac{(x+3)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(x+3)^n}\right| = |x-3|\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = |x-3|\cdot\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^2$

$= |x-3| \Rightarrow R = 3$. And applying the Ratio Test to Series III gives $\lim_{n\to\infty}\left|\frac{n^{n+1}x^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n x^n}\right|$

$= |x|\cdot\lim_{n\to\infty}\left(\frac{1}{n+1}\cdot\frac{n^{n+1}}{n^n}\right) = |x|\cdot\lim_{n\to\infty}\frac{n}{n+1} = |x|\cdot 1 = |x|$, so $R = 1$.

38. If $a_n = \frac{(2x+1)^n}{2^n}$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2x+1)^{n+1}}{2^{n+1}}\cdot\frac{2^n}{(2x+1)^n}\right| = \lim_{n\to\infty}|2x+1|\cdot\frac{1}{2} = \frac{|2x+1|}{2}$. By the Ratio Test the series $\sum_{n=0}^{\infty}\frac{(2x+1)^n}{2^n}$ converges when $\frac{|2x+1|}{2} < 1 \Leftrightarrow |2x+1| < 2 \Leftrightarrow -2 < 2x+1 < 2 \Leftrightarrow$

$-3 < 2x < 1 \Leftrightarrow -\frac{3}{2} < x < \frac{1}{2}$. When $x = \frac{1}{2}$, the series $\sum_{n=0}^{\infty}\frac{(2(\frac{1}{2})+1)^n}{2^n} = \sum_{n=0}^{\infty}\frac{2^n}{2^n} = \sum_{n=0}^{\infty}1$ diverges by the Test for Divergence. When $x = -\frac{3}{2}$, the series $\sum_{n=0}^{\infty}\frac{(2(-\frac{3}{2})+1)^n}{2^n} = \sum_{n=0}^{\infty}\frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty}(-1)^n$ also diverges by the Test for Divergence. Thus, the correct choice is (**C**).

39. If $a_n = \frac{b(x-a)^n}{k^n}$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{b(x-a)^{n+1}}{k^{n+1}}\cdot\frac{k^n}{b(x-a)^n}\right| = \lim_{n\to\infty}\left|\frac{(x-a)}{k}\right|$, so by the Ratio Test the series converges when $\left|\frac{(x-a)}{k}\right| < 1 \Leftrightarrow |x-a| < k$. This is choice (**A**).

40. The series $2(x-2) + 2(x-2)^2 + (x-2)^3 + \frac{(x-2)^4}{3} + \frac{(x-2)^5}{12} + \cdots$

$$= \frac{2\cdot 0}{0!}(x-2)^0 + \frac{2\cdot 1}{1!}(x-2)^1 + \frac{2\cdot 2}{2!}(x-2)^2 + \frac{2\cdot 3}{3!}(x-2)^3 + \frac{2\cdot 4}{4!}(x-2)^4 + \frac{2\cdot 5}{5!}(x-2)^5 + \cdots$$

can be written $\sum_{n=0}^{\infty}\frac{2n(x-2)^n}{n!}$, which is option (**D**).

45. Because the series converges at $x = 7$ and diverges at $x = 10$, we know $2 \le R$, but $R \le 5$. Therefore the series must converge for all $|x-5| < 2$ and must diverge for all $|x-5| > 5$. Therefore, we can only be sure that statement (**B**), the series converges at $x = 4$, is true.

46. $a_n = \sum_{n=0}^{\infty} \frac{(x+2)^n}{(2n+1)^2} \Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(2n+3)^2}\cdot\frac{(2n+1)^2}{(x+2)^n}\right| = \lim_{n\to\infty}|x+2| = |x+2|$. By the Ratio Test, the series converges for $|x+2| < 1 \Leftrightarrow -3 < x < -1$. When $x = -3$, the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^2}$ converges by the Alternating Series Test. When $x = -1$, the series $\sum_{n=1}^{\infty}\frac{1^n}{(2n+1)^2}$ converges by comparison with the convergent p-series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ $(p = 2 > 1)$. So the values of x for which the series $\sum_{n=1}^{\infty}\frac{(x+2)^n}{(2n+1)^2}$ converges are (**A**) $-3 \le x \le -1$.

47. By the Ratio test, the series $\sum_{n=3}^{\infty}\frac{(-1)^{n+1}(x-2)^n}{n\ln n}$ converges for $|x-2| < 1 \Leftrightarrow 1 < x < 3$. When $x = 1$, the series is divergent by Integral Test. When $x = 3$, the series $\sum_{n=3}^{\infty}\frac{(-1)^{n+1}}{n\ln n}$ is convergent by the Alternating Series Test. Therefore $x = 3$ is in the interval of convergence of the power series. This is choice (**C**).