p. 725: 9-12, 19, 21-23, 27-37 odd, 39-48

9. Our goal is to write the function in the form $\dfrac{1}{1-r}$, and the represent the function as a sum of a power series. $f(x)=\dfrac{1}{1+x}=\dfrac{1}{1-(-x)}=\sum_{n=0}^{\infty}(-x)^n\sum_{n=0}^{\infty}(-1)^n x^n$ with $|-x|<1 \Leftrightarrow |x|<1$, so $R=1$ and $I=(-1,1)$.

10. $f(x)=\dfrac{5}{1-4x^2}=5\left(\dfrac{1}{1-4x^2}\right)=5\sum_{n=0}^{\infty}(4x^2)^n=5\sum_{n=0}^{\infty}4^n x^{2n}$. The series converges when $|4x^2|<1 \Leftrightarrow$ $|x|^2<\frac{1}{4} \Leftrightarrow |x|<\frac{1}{2}$, so $R=\frac{1}{2}$ and $I=\left(-\frac{1}{2},\frac{1}{2}\right)$.

11. $f(x)=\dfrac{2}{3-x}=\dfrac{2}{3}\left(\dfrac{1}{1-x/3}\right)=\dfrac{2}{3}\sum_{n=0}^{\infty}\left(\dfrac{x}{3}\right)^n$ or, equivalently, $2\sum_{n=0}^{\infty}\dfrac{1}{3^{n+1}}x^n$. The series converges when $\left|\dfrac{x}{3}\right|<1$, that is, when $|x|<3$, so $R=3$ and $I=(-3,3)$.

12. $f(x)=\dfrac{4}{2x+3}=\dfrac{4}{3}\left(\dfrac{1}{1+2x/3}\right)=\dfrac{4}{3}\left(\dfrac{1}{1-(-2x/3)}\right)=\dfrac{4}{3}\sum_{n=0}^{\infty}\left(-\dfrac{2x}{3}\right)^n$ or, equivalently, $\sum_{n=0}^{\infty}(-1)^n\dfrac{2^{n+2}}{3^{n+1}}x^n$. The series converges when $\left|-\dfrac{2x}{3}\right|<1$, that is, when $|x|<\frac{3}{2}$, so $R=\frac{3}{2}$ and $I=\left(-\frac{3}{2},\frac{3}{2}\right)$.

19. (a) $f(x)=\dfrac{1}{(1+x)^2}=\dfrac{d}{dx}\left(\dfrac{-1}{1+x}\right)=-\dfrac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]=\sum_{n=1}^{\infty}(-1)^{n+1}(n+1)x^{n-1}=\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$ with $R=1$. In the last step, note that we decreased the initial value of the summation variable n by 1, and then increased each occurrence of n in the term by 1. [Also note that $(-1)^{n+2}=(-1)^n$].

(b) $f(x)=\dfrac{1}{(1+x)^3}=-\dfrac{1}{2}\dfrac{d}{dx}\left[\dfrac{1}{(1+x)^2}\right]=-\dfrac{1}{2}\dfrac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n(n+1)x^n\right]$ [from part (a)]

$=-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1}=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ with $R=1$.

(c) $f(x)=\dfrac{x^2}{(1+x)^3}=x^2\cdot\dfrac{1}{(1+x)^3}=x^2\cdot\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ [from part (b)]

$=\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}$.

To write the power series with x^n rather than x^{n+2}, we will decrease each occurrence of n in the term x^{n+2}, by 2 and increase the initial value of the summation variable by 2. This gives us

$\frac{1}{2}\sum_{n=2}^{\infty}(-1)^n(n)(n-1)x^n$ with $R=1$.

21. $f(x)=\ln(5-x)-\int\dfrac{dx}{5-x}=-\frac{1}{5}\int\dfrac{dx}{1-x/5}=-\frac{1}{5}\int\left[\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n\right]dx=C-\frac{1}{5}\sum_{n=0}^{\infty}\dfrac{x^{n+1}}{5^n(n+1)}=C-\sum_{n=1}^{\infty}\dfrac{x^n}{n5^n}$.

Putting $x=0$, we get $C=\ln 5$. The series converges for $|x/5|<1 \Leftrightarrow |x|<5$, so $R=5$.

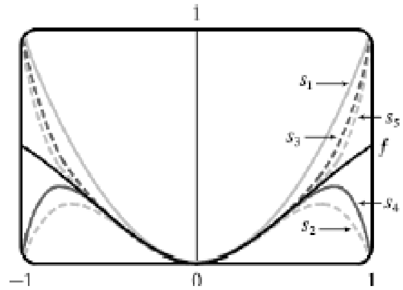
22. $f(x)=x^2\tan^{-1}(x^3)=x^2\sum_{n=0}^{\infty}(-1)^n\dfrac{(x^3)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{6n+3+2}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{6n+5}}{2n+1}$ for $|x^3|<1 \Leftrightarrow |x|<1$, so $R=1$.

23. We know that $\dfrac{1}{1+4x} = \dfrac{1}{1-(-4x)} = \sum_{n=0}^{\infty} (-4x)^n$. Differentiating, we get

$$\frac{-4}{(1+4x)^2} = \sum_{n=1}^{\infty} (-4)^n n x^{n-1} = \sum_{n=0}^{\infty} (-4)^{n+1}(n+1)x^n, \text{ so } f(x) = \frac{x}{(1+4x)^2} = \frac{-x}{x} \cdot \frac{-4}{(1+4x)^2}$$

$$= \frac{-x}{4} \sum_{n=0}^{\infty} (-4)^{n+1}(n+1)x^n = \sum_{n=0}^{\infty} (-1)^n 4^n (n+1) x^{n+1} \quad |-4x| < 1 \Leftrightarrow |x| < \tfrac{1}{4}, \text{ so } R = \tfrac{1}{4}.$$

27. $f(x) = \dfrac{x^2}{x^2+1} = x^2\left(\dfrac{1}{1-(-x^2)}\right) = x^2 \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n+2}$. The series converges when $|x^2| < 1 \Leftrightarrow x^2 < 1 \Leftrightarrow |x| < 1$, so $R = 1$. The partial sums are $s_1 = x^2$, $s_2 = s_1 - x^4$, $s_3 = s_2 + x^6$, $s_4 = s_3 - x^8$, $s_5 = s_4 + x^{10}, \cdots$. Note that s_1 corresponds to the first term of the infinite sum, regardless of the value of the summation variable and the value of the exponent. As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-1,1)$.

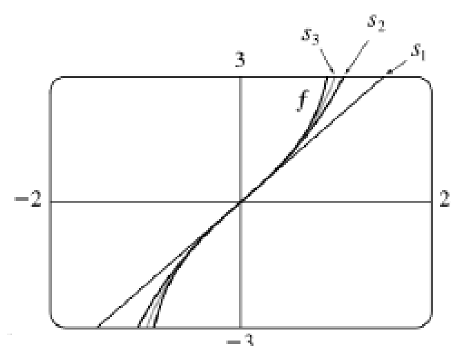


29. $f(x) = \ln\left(\dfrac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \displaystyle\int \frac{dx}{1+x} + \int \frac{dx}{1-x} = \int \frac{dx}{1-(-x)} + \int \frac{dx}{1-x}$

$$= \int \left[\sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n \right] dx = \int \left[(1 - x + x^2 - x^3 + x^4 - \cdots) + (1 + x + x^2 + x^3 + x^4 - \cdots) \right] dx$$

$$= \int (2 + 2x^2 + 2x^4 + \cdots)\, dx = \int \sum_{n=0}^{\infty} 2x^{2n}\, dx = C + \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}.$$

But $f(0) = \ln \frac{1}{1} = 0$ so $C = 0$ and we have $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$ with $R = 1$. If $x = \pm 1$, then $f(x) = \pm 2 \displaystyle\sum_{n=0}^{\infty} \frac{1}{2n+1}$, which both diverge by the



Limit Comparison Test with $b_n = \dfrac{1}{n}$. The partial sums are $s_1 = \dfrac{2x}{1}, s_2 = s_1 - \dfrac{2x^3}{3}, s_3 = s_2 - \dfrac{2x^5}{5}, \cdots$. As n increases, $s_n(x)$ approximates f better on the interval of convergence which is $(-1,1)$.

31. $\dfrac{t}{1-t^8} = t \cdot \dfrac{1}{1-t^8} = t \displaystyle\sum_{n=0}^{\infty} (t^8)^n = \sum_{n=0}^{\infty} t^{8n+1} \Leftrightarrow \int \frac{t}{1-t^8}\, dt = C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}$. The series for $\dfrac{t}{1-t^8}$ converges when $|t^8| < 1 \Leftrightarrow |t| < 1$, so $R = 1$ for that series and also the series for $t/(1-t^8)$. By Theorem 1, the series for $\displaystyle\int \frac{t}{1-t^8}\, dt$ also has $R = 1$.

33. $\ln(1+x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ for $|x| < 1$, so $x^2 \ln(1+x) = \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{n+2}}{n}$ and

$\displaystyle\int x^2 \ln(1+x)\, dx = C + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{n+3}}{n(n+3)}$. $R = 1$ for the series for $\ln(1+x)$, so $R = 1$ so for the series representing $x^2 \ln(1+x)$ as well. By Theorem 1, the series for $\int x^2 \ln(1+x)\, dx$ also has $R = 1$.

35. $\frac{x}{1+x^3}=x\left[\frac{1}{1-(-x^3)}\right]=x\sum_{n=0}^{\infty}(-x^3)^n=\sum_{n=0}^{\infty}(-1)^n x^{3n+1} \Rightarrow$

$\int\frac{x}{1+x^3}dx=\int\sum_{n=0}^{\infty}(-1)^n x^{3n+1}dx=C+\sum_{n=0}^{\infty}(-1)^n\frac{x^{3n+2}}{3n+2}$. Thus,

$I\approx\int_0^{0.3}\frac{x}{1+x^3}dx=\left[\frac{x^2}{2}-\frac{x^5}{5}+\frac{x^8}{8}-\frac{x^{11}}{11}+\cdots\right]_0^{0.3}=\frac{(0.3)^4}{2}-\frac{(0.3)^5}{5}+\frac{(0.3)^8}{8}-\frac{(0.3)^{11}}{11}+\cdots$. The series is alternating, so if we use the first three terms, the error is at most $(0.3)^{11}/11\approx1.6\times10^{-7}$. So $I\approx(0.3)^2/2-(0.3)^5/5+(0.3)^8/8\approx0.044522$ to six decimal places.

37. We substitute x^2 for x in Example 5 and find that

$\int x\ln(1+x^2)\,dx=\int x\sum_{n=1}^{\infty}(-1)^n\frac{(x^2)^n}{n}dx=\int\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n+1}}{n}dx=C+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n+2}}{n(2n+2)}$. Thus,

$I\approx\int_0^{0.2}x\ln(1+x^2)\,dx=\left[\frac{x^4}{1(4)}-\frac{x^6}{2(6)}+\frac{x^8}{3(8)}-\frac{x^{10}}{4(10)}+\cdots\right]_0^{0.2}=\frac{(0.2)^4}{4}-\frac{(0.2)^6}{12}+\frac{(0.2)^8}{24}-\frac{(0.2)^{10}}{40}+\cdots$.

The series is alternating, so if we use two terms, the error is at most $(0.2)^8/24\approx1.1\times10^{-7}$. So $I\approx\frac{(0.2)^4}{4}-\frac{(0.2)^6}{12}\approx0.000395$ to six decimal places.

39. $\frac{1}{1+x^3}=\frac{1}{1-(-x^3)}=1+(-x^3)+(-x^3)^2+(-x^3)^3+\cdots=1-x^3+x^6-x^9+\cdots$ so choice(**D**) is correct.

40. $\frac{1}{1-x^4}=1+(x^4)^1+(x^4)^2+(x^4)^3\cdots+\cdots=1+x^4+x^8+x^{12}+\cdots$.

$\int\frac{dx}{1-x^4}=\int\left(1+x^4+x^8+x^{12}+\cdots\right)dx=x+\frac{1}{5}x^5+\frac{1}{9}x^9+\frac{1}{13}x^{13}+\cdots+C$, so the correct choice is (**B**).

41. (a) $S(x)=6\cdot\sum_{n=1}^{\infty}x^n$ which converges when $|x|<1\Leftrightarrow -1<x<1$. This means that 5 is not in the interval of convergence, so $S(5)$ does not exist.

(b) 0.5 is in the interval of convergence for this power series, so $S(0.5)$ does exist. In fact, $S(0.5)=6\cdot\sum_{n=1}^{\infty}(0.5)^n=6\left(\frac{1}{1-0.5}-1\right)=6(2-1)=6.$

(c) The domain of this power series is the set of all real numbers for which the power series converges. We have already seen that the radius of convergence is $R=1$. When $x=1$, the power series is $S(1)=6\cdot\sum_{n=1}^{\infty}1$, which diverges by the Test for Divergence. When $x=-1$, the series is $S(-1)=6\cdot\sum_{n=1}^{\infty}(-1)^n=-6+6-6+6-6+\cdots$, which also diverges. Thus, the domain of S is $(-1,1)$.

(d) The function S is a geometric series with $a=6x$ and $r=x$ so it is equivalent to $f(x)=\frac{6x}{1-x}$ over the interval $(-1,1)$.

(e) $S(0.1)=6\cdot\sum_{n=1}^{\infty}(0.1)^n=\dfrac{6(0.1)}{(1-0.1)}=0.\overline{6};\ f(0.1)=\dfrac{0.6}{0.9}=0.\overline{6};\ \left|f(0.1)-S_n(0.1)\right|$ is the error in approximating by using the first n terms of the power series S(x).

42. (a) $s(x)=4x^2\left(1+(2x^2)+(2x^2)^2+(2x^2)^3+(2x^2)^4+\cdots\right)=4x^2+8x^4+16x^6+32x^8+\cdots\sum_{n=1}^{\infty}2^{n+1}x^{2n}$.

(b) $s(x)$ converges when $\left|-2x^2\right|<1\Leftrightarrow -1<2x^2<1\Leftrightarrow -\frac{1}{2}<x^2<\frac{1}{2}\Leftrightarrow -\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$, so $R=\frac{1}{\sqrt{2}}$.

When $x=\pm\frac{1}{\sqrt{2}}$, the series is $\sum_{n=1}^{\infty}2^{n+1}\left(\pm\frac{1}{\sqrt{2}}\right)^{2n}=\sum_{n=1}^{\infty}2^{n+1}\left(\frac{1}{2}\right)^n=\sum_{n=1}^{\infty}\dfrac{2^{n+1}}{2^n}=\sum_{n=1}^{\infty}2$, which diverges by the nth term test. Thus, the interval of convergence for $s(x)$ is $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$.

43. (a) $s_4(x)=\frac{1}{2}x+x^2+2x^3+4x^4$; The general term is $2^{n-1}\cdot x^{n+1}$.

(b) $s(x)=\sum_{n=0}^{\infty}2^{n-1}\cdot x^{n+1}\ \Leftrightarrow\ s(1)=\sum_{n=0}^{\infty}2^{n-1}\cdot 1^{n+1}=\infty$. Clearly $x=1$ is not in the interval of convergence for this power series.

(c) By the integral test, $s(x)$ converges when $|2x|<1\Leftrightarrow -1<2x<1\Leftrightarrow -\frac{1}{2}<x<\frac{1}{2}$, so $R=\frac{1}{2}$. When $x=\frac{1}{2}$, the series is $s\left(\frac{1}{2}\right)=\sum_{n=0}^{\infty}2^{n-1}\cdot\left(\frac{1}{2}\right)^{n+1}=\sum_{n=0}^{\infty}\dfrac{2^{n-1}}{2^{n+1}}=\sum_{n=0}^{\infty}\dfrac{1}{2^2}$, which diverges by the nth Term Test.

When the series is $s\left(-\frac{1}{2}\right)=\sum_{n=0}^{\infty}2^{n-1}\cdot\left(-\frac{1}{2}\right)^{n+1}=\sum_{n=0}^{\infty}\dfrac{2^{n-1}}{2^{n+1}}=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{4}=-\frac{1}{4}+\frac{1}{4}-\frac{1}{4}+\frac{1}{4}-\cdots$, which does not converge. Thus the interval of convergence for this power series is $\left(-\frac{1}{2},\frac{1}{2}\right)$.

(d) $\dfrac{1}{1-2x}=1+2x+(2x)^2+(2x)^3+(2x)^4+\cdots\Rightarrow\dfrac{1}{2(1-2x)}=\frac{1}{2}+x+2x^2+4x^3+8x^4+\cdots\Rightarrow$
$\dfrac{2}{2(1-2x)}=\frac{1}{2}x+x^2+2x^3+4x^4+8x^5+\cdots=s(x)$. Thus, $f(x)=\dfrac{x}{2(1-2x)}$.

44. (a) $\dfrac{1}{1+x}=1-x+x^2-x^3+x^4-\cdots$

(b) $\ln(1+x)=C+x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots$

(c) The first four terms of the power series for $\ln(1+x^3)$ are $x^3,-\frac{1}{2}x^6,+\frac{1}{3}x^9$, and $-\frac{1}{4}x^{12}$. The general term is $(-1)^{n+1}\frac{1}{n}x^{3n}$.

45. (a) Assuming 0.8 is in the interval of convergence, the Alternating Series Error Bound says that if the first four terms of the series are used to approximate $f(0.8)$, then the error in that approximation will be $|R_4|\le\dfrac{(0.8)^5}{5}=0.065536<0.1$.

(b) For $x=0.8$, the series is $f(0.8)=\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{(0.8)^n}{n}=\ln(1.8)=5.877\overline{86}$.

46. (a) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{(n+1)\cdot 3^{n+1}}\cdot\frac{n\cdot 3^n}{(x-3)^n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{3}\cdot\frac{n}{n+1}\right| = \frac{1}{3}|x-3|$. The series converges for $\frac{1}{3}|x-3|<1 \Rightarrow |x-3|<3 \Rightarrow -3<x-3<3 \Rightarrow 0<x<6$. When $x=6$, the series is $\sum_{n=1}^{\infty}\frac{3^n}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{1}{n}$ which is the divergent harmonic series. When $x=0$, the series is $\sum_{n=1}^{\infty}\frac{(-3)^n}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ which is the convergent alternating harmonic series. Thus, the interval of convergence for this power series is $[0,6)$.

(b) $f'(x) = \frac{1}{3} - \frac{(x-3)}{3^2} + \frac{(x-3)^2}{3^3} - \frac{(x-3)^3}{3^4} + \cdots \frac{(-1)^{n+1}(x-3)^{n-1}}{3^n} + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-3)^{n-1}}{3^n}$, so the first 3 nonzero terms are $\frac{1}{3}, -\frac{(x-3)}{3^2}$, and $\frac{(x-3)^2}{3^3}$, and the general term is $(-1)^{n+1}\frac{(x-3)^{n-1}}{3^n}$.

(c) The power series for $f'(x)$ found in part (c) is geometric with $a=\frac{1}{3}$, and $r=-\frac{(x-3)}{3}$. Therefore, this series converges to $f'(x) = \frac{a}{1-r} = \frac{\frac{1}{3}}{1+\frac{(x-3)}{3}} = \frac{\frac{1}{3}}{\frac{3+x-3}{3}} = \frac{1}{x}$.

(d) We are given that $f(x) = \frac{(x-3)}{3} - \frac{((x-3)/3)^2}{2} + \frac{((x-3)/3)^3}{3} - \frac{((x-3)/3)^4}{4} + \cdots$, and by substitution, $\frac{(x-3)}{3} - \frac{((x-3)/3)^2}{2} + \frac{((x-3)/3)^3}{3} - \frac{((x-3)/3)^4}{4} + \cdots = \ln\left(1+\frac{x-3}{3}\right)$.

Thus, $f(x) = \ln\left(1+\frac{x-3}{3}\right) = \ln\left(\frac{3+x-3}{3}\right) = \ln\left(\frac{x}{3}\right) = \ln(x) - \ln 3 \Rightarrow k = -\ln 3$.

47. The interval of convergence of a power series must be symmetric about its center, so $-1<x<5$ is the interval of convergence of $\sum_{n=0}^{\infty}a_n(x-k)^n$, then k must be 2. In addition, because the ratio of convergence is 3, we must have $3 = \frac{a_0}{a_1} = \frac{18}{a_1} \Rightarrow a_1 = 6$. Therefore, the correct choice is (**A**).

48. If $f(x) = \sum_{n=1}^{\infty}\frac{2^n x^n}{n}$, then $f\left(-\frac{1}{4}\right) = \sum_{n=1}^{\infty}\frac{2^n\left(-\frac{1}{4}\right)^n}{n} = \sum_{n=1}^{\infty}\frac{\left(-\frac{1}{2}\right)^n}{n} = \frac{\left(-\frac{1}{2}\right)^1}{1} + \frac{\left(-\frac{1}{2}\right)^2}{2} + \frac{\left(-\frac{1}{2}\right)^3}{3} + \frac{\left(-\frac{1}{2}\right)^4}{4} + \cdots$

$= \left[-\frac{\left(\frac{1}{2}\right)^1}{1} + \frac{\left(\frac{1}{2}\right)^2}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^4}{4} + \cdots\right] = -\left[\frac{\left(\frac{1}{2}\right)^1}{1} - \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{3} - \frac{\left(\frac{1}{2}\right)^4}{4} + \cdots\right] = -\ln\left(1+\frac{1}{2}\right) = -\ln\left(\frac{3}{2}\right) = \ln\left(\frac{2}{3}\right)$,

option (**B**).