## Error in Taylor Series

p. 745: 45-51 odd, 86-88, 97-99, 101, 103, 109-115 odd, 117-118, 122-132

45. (a)

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(4)$ |
|---|---|---|
| 0 | $x^{-1/2}$ | $\frac{1}{2}$ |
| 1 | $-\frac{1}{2}x^{-3/2}$ | $-\frac{1}{16}$ |
| 2 | $\frac{3}{4}x^{-5/2}$ | $\frac{3}{128}$ |
| 3 | $-\frac{15}{8}x^{-7/2}$ | |

$f(x) = x^{-1/2} \approx T_2(x)$

$$T_2(x) = \frac{1/2}{0!}(x-4)^0 - \frac{1/16}{1!}(x-4)^1 + \frac{3/128}{2!}(x-4)^2$$

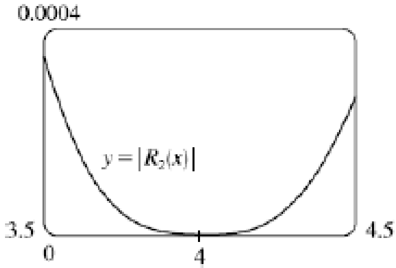
$$= \tfrac{1}{2} - \tfrac{1}{16}(x-4) + \tfrac{3}{256}(x-4)^2$$

(b) $|R_2(x)| \le \frac{M}{3!}|x-4|^3$, where $|f'''(x)| \le M$. Now $3.5 \le x \le 4.5 \Rightarrow |x-4| \le 0.5 \Rightarrow |x-4|^3 \le 0.125$.

Since $|f'''(x)|$ is decreasing on $[3.5, 4.5]$, we can take $M = |f'''(3.5)| = \frac{15}{8(3.5)^{7/2}}$, so

$$|R_2(x)| \le \frac{15}{6 \cdot 8(3.5)^{7/2}}(0.125) \approx 0.000487.$$

(c)




From the graph of $|R_2(x)| = |x^{-1/2} - T_2(x)|$, it seems that the error is less than 0.000343 on $[3.5, 4.5]$.

47. (a)

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(1)$ |
|---|---|---|
| 0 | $\sin x$ | $\frac{1}{2}$ |
| 1 | $\cos x$ | $\frac{\sqrt{3}}{2}$ |
| 2 | $-\sin x$ | $-\frac{1}{2}$ |
| 3 | $-\cos x$ | $-\frac{\sqrt{3}}{2}$ |
| 4 | $\sin x$ | $\frac{1}{2}$ |
| 5 | $\cos x$ | |

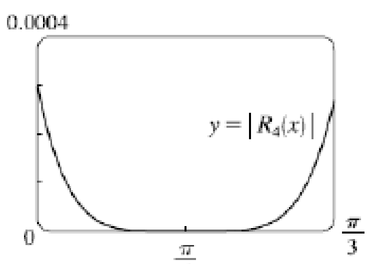
$f(x) = \sin x \approx T_4(x)$

$$T_4(x) = \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\left(x - \tfrac{\pi}{6}\right) - \tfrac{1}{4}\left(x - \tfrac{\pi}{6}\right)^2 - \tfrac{\sqrt{3}}{12}\left(x - \tfrac{\pi}{6}\right)^3 + \tfrac{1}{48}\left(x - \tfrac{\pi}{6}\right)^4$$

(b) $|R_4(x)| \le \frac{M}{5!}\left|x - \frac{\pi}{6}\right|^5$, where $|f^{(5)}(x)| \le M$. Now $1 \le x \le \frac{\pi}{3} \Rightarrow -\frac{\pi}{6} \le x - \frac{\pi}{6} \le \frac{\pi}{6} \Rightarrow \left|x - \frac{\pi}{6}\right| \le \frac{\pi}{6} \Rightarrow$ $\left|x - \frac{\pi}{6}\right|^5 \le \left(\frac{\pi}{6}\right)^5$. Since $|f^{(5)}(x)|$ is decreasing on $\left[0, \frac{\pi}{3}\right]$, we can take $M = |f^{(5)}(0)| = \cos 0 = 1$, so

$$|R_4(x)| \le \frac{1}{5!}\left(\frac{\pi}{6}\right)^5 \approx 0.00328.$$

(c)




From the graph of $|R_4(x)| = |\sin x - T_4(x)|$, it seems that the error is less than 0.000297 on $\left[0, \frac{\pi}{3}\right]$.

49. (a)

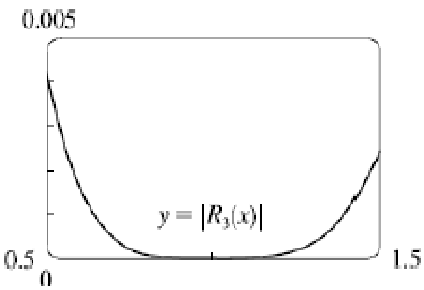
| $n$ | $f^{(n)}(x)$ | $f^{(n)}(1)$ |
|---|---|---|
| 0 | $\ln(1+2x)$ | $\ln 3$ |
| 1 | $2/(1+2x)$ | $2/3$ |
| 2 | $-4/(1+2x)^2$ | $-4/9$ |
| 3 | $16/(1+2x)^3$ | $16/27$ |
| 4 | $-96/(1+2x)^4$ | |

$f(x)=\ln(1+2x)\approx T_3(x)$

$T_3(x)=\ln 3+\frac{2}{3}(x-1)-\frac{2}{9}(x-1)^2+\frac{8}{81}(x-1)^3$

(b) $|R_3(x)|\le\dfrac{M}{4!}|x-4|^4$, where $|f^{(4)}(x)|\le M$. Now $0.5\le x\le 1.5\Rightarrow -0.5\le x-1\le 0.5\Rightarrow$ $|x-1|\le 0.5\Rightarrow$ $|x-1|^4\le\frac{1}{16}$, and letting $x=0.5$ gives $M=6$, $|R_3(x)|\le\dfrac{6}{4!}\cdot\dfrac{1}{16}=\dfrac{1}{64}=0.015625.$

(c)




From the graph of $|R_3(x)|=|\ln(1+2x)-T_3(x)|$, it seems that the error is less than 0.005 on $[0.5,1.5]$.

51. (a)

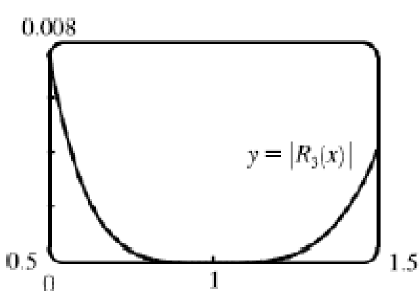
| $n$ | $f^{(n)}(x)$ | $f^{(n)}(1)$ |
|---|---|---|
| 0 | $\ln x$ | 0 |
| 1 | $\ln x+1$ | 1 |
| 2 | $1/x$ | 1 |
| 3 | $-1/x^2$ | $-1$ |
| 4 | $2/x^3$ | |

$f(x)=x\ln x\approx T_3(x)$

$T_3(x)=(x-1)+\frac{1}{2}(x-1)^2-\frac{1}{6}(x-1)^3$ #20

(b) $|R_3(x)|\le\dfrac{M}{4!}|x-1|^4$, where $|f^{(4)}(x)|\le M$. Now $0.5\le x\le 1.5\Rightarrow$ $|x-1|\le\frac{1}{2}\Rightarrow$ $|x-1|^4\le\frac{1}{16}$. Since $|f^{(4)}(x)|$ is decreasing on $[0.5,1.5]$, we can take $M=|f^{(4)}(0.4)|=2/(0.5)^3=16$, so $|R_3(x)|\le\frac{16}{24}(1/16)=\frac{1}{24}=0.041\overline{6}.$

(c)




From the graph $|R_3(x)|=|x\ln x-T_3(x)|$, of it appears that the error is less than 0.0076 on $[0.5,1.5]$.

86. $f^{(4)}(\sin x)=\sin x$ and $|\sin(x)|\le 1$ for all $x$. Thus, $\sin(0.1)\approx(0.1)-\frac{1}{6}(0.1)^3$ will have error $|R_3(x)|\le\dfrac{1}{24}(0.1)^4\approx 0.000004$, choice **(A)**.

87. $f(x)=e^{x-1} \Rightarrow f^{(n)}(x)=e^{x-1}$. Since is $f$ increasing, the maximum value on $[0.8,1.2]$ is $e^{0.2}\approx 1.22140$, and $0.8\le x\le 1.2 \Leftrightarrow |x-1|<0.2$. So $|R_2(x)|=|T_2(x)-f(x)|\le \frac{e^{0.2}}{6}(0.2)^3 \approx 0.00163$. Choice (**B**) is closest to this error bound.

88. $f(x)=\ln(1+x) \Rightarrow f'''(x)=\frac{2}{(1+x)^3}$, which is decreasing and $\le 2$ on $[0,\infty)$. So for $|x|\le 0.2$, $|R_2(x)|=|T_2(x)-f(x)|\le 2\cdot\frac{(0.2)^3}{6}\approx 0.00267<0.005$, choice (**C**).

97. $$\lim_{x\to 0}\frac{x-\ln(1+x)}{x^2}=\lim_{x\to 0}\frac{x-(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5-\cdots)}{x^2}=\lim_{x\to 0}\frac{\frac{1}{2}x^2-\frac{1}{3}x^3+\frac{1}{4}x^4-\frac{1}{5}x^5+\cdots}{x^2}$$
$$=\lim_{x\to 0}(\tfrac{1}{2}-\tfrac{1}{3}x+\tfrac{1}{4}x^2-\tfrac{1}{5}x^3+\cdots)=\tfrac{1}{2}$$

98. $$\lim_{x\to 0}\frac{1-\cos x}{1+x-e^x}=\lim_{x\to 0}\frac{1-(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots)}{1+x-(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\cdots)}$$
$$=\lim_{x\to 0}\frac{\frac{1}{2!}x^2-\frac{1}{4!}x^4+\frac{1}{6!}x^6-\cdots}{-\frac{1}{2!}x^2-\frac{1}{3!}x^3-\frac{1}{4!}x^4-\frac{1}{5!}x^5-\frac{1}{6!}x^6-\cdots}$$
$$=\lim_{x\to 0}\frac{\frac{1}{2!}-\frac{1}{4!}x^2+\frac{1}{6!}x^4-\cdots}{-\frac{1}{2!}-\frac{1}{3!}x-\frac{1}{4!}x^2-\frac{1}{5!}x^3-\frac{1}{6!}x^4-\cdots}=\frac{\frac{1}{2}-0}{-\frac{1}{2}-0}=-1.$$

99. $$\lim_{x\to 0}\frac{\sin x-x+\frac{1}{6}x^3}{x^5}=\lim_{x\to 0}\frac{(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots)-x+\frac{1}{6}x^3}{x^5}$$
$$=\lim_{x\to 0}\frac{\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots}{x^5}=\lim_{x\to 0}\left(\frac{1}{5!}-\frac{x^2}{7!}+\frac{x^4}{9!}-\cdots\right)=\frac{1}{5!}=\frac{1}{120}.$$

101. $$\lim_{x\to 0}\frac{x^3-3x+3\tan^{-1}x}{x^5}=\lim_{x\to 0}\frac{x^3-3x+3(x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\cdots)}{x^5}$$
$$=\lim_{x\to 0}\frac{x^3-3x+3x-x^3+\frac{3}{5}x^5-\frac{3}{7}x^7+\cdots}{x^5}=\lim_{x\to 0}\frac{\frac{3}{5}x^5-\frac{3}{7}x^7+\cdots}{x^5}$$
$$=\lim_{x\to 0}\left(\tfrac{3}{5}-\tfrac{3}{7}x^2+\cdots\right)=\tfrac{3}{5}$$ since power functions are continuous.

103. We know $e^{-x^2}=1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots$ and $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$. Therefore, $e^{-x^2}\cos x=(1-x^2+\frac{1}{2}x^4-\cdots)(1-\frac{1}{2}x^2+\frac{1}{24}x^4-\cdots)$. Writing only terms with degree $\le 4$, we get $e^{-x^2}\cos x=1-\frac{1}{2}x^2+\frac{1}{24}x^4-x^2+\frac{1}{4}x^4+\frac{1}{2}x^4+\cdots=1-\frac{3}{2}x^2+\frac{25}{24}x^4+\cdots$

109. $$\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{n!}=\sum_{n=0}^{\infty}\frac{(-x^4)^n}{n!}=e^{-x^4}$$

111. $$\sum_{n=1}^{\infty}(-1)^n\frac{3^n}{n5^n}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(3/5)^n}{n}=\ln\left(1+\tfrac{3}{5}\right)=\ln\left(\tfrac{8}{5}\right)$$

113. $$\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{4^{2n+1}(2n+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!}=\sin\tfrac{\pi}{4}=\tfrac{1}{\sqrt{2}}$$

115. $3+\frac{9}{2!}+\frac{27}{3!}+\frac{81}{4!}+\cdots=\frac{3^1}{1!}+\frac{3^2}{2!}+\frac{3^3}{3!}+\frac{3^4}{4!}+\cdots=\sum_{n=1}^{\infty}\frac{3^n}{n!}=\sum_{n=0}^{\infty}\frac{3^n}{n!}-1=e^3-1$

117. (a) $T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(1)}{n!}(x-1)^n=2+3(x-1)+2(x-1)^2+\frac{5}{6}(x-1)^3.$

(b) $f(1.5)\approx T_3(1.5)=4.1041\overline{6}$

(c) Using the Lagrange Error Bound, $\left|T_3(1.5)-f(1.5)\right|\le\frac{M}{4!}\left|1.5-1\right|^4$ where $M\le\left|f^{(4)}(x)\right|$. We know $M\le 3$, so $\left|T_3(1.5)-f(1.5)\right|\le 3\cdot\frac{0.5^4}{24}=0.0078125<0.01.$

118. (a) $T_3(x)=5+12(x-1)-3(x-1)^2+18(x-1)^3\Rightarrow T_1(x)=5+12(x-1),$ so $f(2)\approx T_1(2)=17.$ The tangent line approximation of $f$ at for $x$ near 1 is $f(1)+f'(1)(x-1)=T_1(x).$

(b) From $T_3(x)$ we find that $\frac{f''(2)}{2}=-3\Rightarrow f''(2)=-6,$ so $f$ is concave down at $x=2$. But we are told that $f''$ does not change signs for $x\ge 1$, so the tangent line approximation overestimates, and thus $f(2)<17$.

(c) $T_4(x)=T_3(x)+\frac{f^{(4)}(1)}{4!}(x-1)^4=T_3(x)+\frac{312}{24}(x-1)^4$

$=5+12(x-1)-3(x-1)^2+18(x-1)^3+13(x-1)^4$

$T_4(2)=5+12-3+18+13=45.$

(d) Assume $f(2)=70$. Then $\left|R_4(2)\right|=\left|f(2)-T_4(2)\right|=\left|70-45\right|=25,$ but this contradicts the Lagrange error bound which says $\left|R_4(2)\right|\le\frac{M}{5!}\left|x-1\right|^5$, where $\left|f^{(5)}(x)\right|\le M$. We are given $\left|f^{(5)}(x)\right|\le 1440$ on $1\le x\le 2$, so $\left|R_4(2)\right|\le\frac{1440\cdot 1^5}{5!}=12<25$. Therefore, $f(2)\ne 70$.

122. (a) $T_1(x)=g(2)+g'(2)(x-2)=5+6(x-2)=6x-7.$ $g(3)\approx T_1(3)=6(3)-7=11.$

(b) $\int_2^3 g'(x)\,dx\approx g'(2)\cdot\frac{1}{2}+g'(2.5)\cdot\frac{1}{2}=\frac{1}{2}[6+10]=8\approx g(3)$

(c) $T_3(x)=\sum_{n=0}^{n}\frac{f^{(n)}(2)}{n!}(x-2)^n=\frac{5}{0!}(x-2)^0+\frac{6}{1!}(x-2)^1+\frac{28}{2!}(x-2)^2+\frac{48}{3!}(x-2)^3.$

$=5+6(x-2)+14(x-2)^2+8(x-2)^3.$

$g(3)\approx T_3(3)=5+6(1)+14(1)^2+8(1)=33$

(d) Assume $g(3)=40$. Then $\left|R_3(3)\right|=\left|g(3)-T_3(3)\right|=\left|40-33\right|=7,$ but this contradicts the Lagrange error bound which says $\left|R_3(3)\right|\le\frac{M}{4!}\left|x-2\right|^4$, where $\left|g^{(4)}(x)\right|\le M$. We are given $\left|g^{(4)}(x)\right|\le 80$ on $2\le x\le 3$, so $\left|R_3(3)\right|\le\frac{80\cdot 1^4}{4!}=\frac{10}{3}$. Therefore, $g(3)\ne 40$.

123. (a) $T_2(x)=\sum_{n=0}^{2}\frac{f^{(n)}(5)}{n!}(x-5)^n=\frac{f(5)}{0!}(x-5)^0+\frac{f'(5)}{1!}(x-5)^1+\frac{f''(5)}{2!}(x-5)^2$

$=3+4(x-5)+10(x-5)^2$

(b) $f(5.4)\approx T_2(5.4)=3+4(0.4)+10(0.4)^2=6.2$

(c) $|R_2(5.4)|=|f(5.4)-T_2(5.4)|\le\frac{M}{3!}|x-5|^3$, where $|f^{(3)}(x)|\le M$. We are given $|f^{(3)}(x)|<7$ on $5\le x\le 5.4$, so $|R_2(5.4)|<\frac{7}{6}\cdot(0.4)^3=0.074\overline{6}$.

124. On $0\le x\le 1$, $|g^{(5)}(x)|=|e^{\cos x}|\le e^1=e$. Then the largest possible value for the error used in approximating $g$ at a value $0\le a\le 1$ is $\frac{e}{5!}|1|^5=\frac{e}{5!}\approx 0.02265$, so among the given choices, (**B**) is the largest possible value.

125. (a) $f'(3)=a_1=0$, so $f$ has a critical point at $x=3$. $\frac{f''(3)}{2}=-9\Leftrightarrow f''(3)=-18<0$, so $f$ has a relative maximum at $x=3$.

(b) $f(1)\approx T_3(1)=7-9(-2)^2-3(-2)^3=-5$.

(c) The Lagrange error bound says $|R_3(1)|\le\frac{M}{4!}|1-3|^4$, where $M\le|f^{(5)}(x)|\le 5$. So $|R_3(1)|\le\frac{5\cdot 2^4}{24}=\frac{10}{3}$. But $|R_3(1)|=|f(1)-T_3(1)|=|f(1)+5|\le\frac{10}{3}\Rightarrow-\frac{10}{3}\le f(1)+5\le\frac{10}{3}\Rightarrow$ $-\frac{25}{3}<f(1)<-\frac{5}{3}\Rightarrow f(1)<0$.

126. (a) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{2^{n+1}(x-1)^n}{(n+1)}\cdot\frac{n}{2^n(x-1)^n}\right|=\lim_{n\to\infty}|2(x-1)|\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)=|2(x-1)|$ for convergence, $|x-1|<\frac{1}{2}\Leftrightarrow-\frac{1}{2}<x-1<\frac{1}{2}\Leftrightarrow\frac{1}{2}<x<\frac{3}{2}$. When $x=\frac{3}{2}$, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^n\left(\frac{1}{2}\right)^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is Alternating harmonic series, and so converges by the Alternating Series Test. When $x=\frac{1}{2}$, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^n\left(-\frac{1}{2}\right)^n}{n}=\sum_{n=1}^{\infty}\frac{-1}{n}=-\sum_{n=1}^{\infty}\frac{1}{n}$ is the negative of the harmonic series, and so diverges. Therefore, the interval of convergence is $I=\left(\frac{1}{2},\frac{3}{2}\right]$.

(b) $f(x)=\frac{2}{1}(x-1)^3(x-1)-\frac{4}{1}(x-1)^3(x-1)^2+\frac{8}{3}(x-1)^3-\frac{16}{4}(x-1)^4+\frac{32}{5}(x-1)^5-\cdots$

$\frac{f^{(n)}(1)}{n!}=a_n\Leftrightarrow f^{(n)}(1)=a_n\cdot n!$ $f^{(1)}(1)=2$, $f^{(2)}(1)=-2^2\cdot 1!$, $f^{(3)}(1)=\frac{2^3}{3}\cdot 3!=2^3\cdot 2!$,

$f^{(4)}(1)=-\frac{2^4}{4}\cdot 4!=2^4\cdot 3!,\cdots\Rightarrow f^{(n)}(1)=(-1)^{n-1}\cdot 2^n\cdot(n-1)!$

(c) $f\left(\frac{1}{2}x+\frac{1}{2}\right)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}\left(\frac{1}{2}x+\frac{1}{2}-1\right)^n=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}\left(\frac{1}{2}\right)^n(x-1)^n=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}$

(d) The first three terms of the Taylor series for $f(x)$ are $P_3(x)=2(x-1)-2(x-1)^2+\frac{8}{3}(x-1)^3$

We can approximate $-\ln 2=f\left(\frac{3}{4}\right)\approx P_3\left(\frac{3}{4}\right)=2\left(-\frac{1}{4}\right)-2\left(-\frac{1}{4}\right)^2+\frac{8}{3}\left(-\frac{1}{4}\right)^3=-\frac{7}{12}=-0.58\overline{3}$

(e) $P_3'(x)=2-4(x-1)+8(x-1)^2$. $f'(x)=\sum_{n=0}^{\infty}(-1)^n 2^{n+1}(x-1)^n$

(f) The fourth-degree Taylor polynomial for $g$ centered at 1 is

$P_g(x)=\int P_3(x)\,dx = C+(x-1)^2-\frac{2}{3}(x-1)^3+\frac{2}{9}(x-1)^4$. Then $g(1)=2 \Rightarrow C=2$, so

$P_g(x)=2+(x-1)^2-\frac{2}{3}(x-1)^3+\frac{8}{9}(x-1)^4$

127. (a) $\frac{1}{(1+x)}=\frac{1}{(1-(-x))}=1-x+x^2-x^3+x^4-\cdots=\sum_{n=0}^{\infty}(-x)^n \Rightarrow$

$\frac{-1}{(1+x)^2}=-1+2x-3x^2+4x^3-\cdots=\sum_{n=0}^{\infty}-n(-x)^{n-1}=\sum_{n=0}^{\infty}(-1)^n nx^{n-1} \Rightarrow$

$\frac{2}{(1+x)^3}=2-6x+12x^2-\cdots=\sum_{n=0}^{\infty}(-1)^n n(n-1)x^{n-2} \Rightarrow$

$\frac{1}{(1+x)^3}=\frac{1}{2}\left[2-6x+12x^2-\cdots\right]=1-3x+6x^2-\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{n(n-1)}{2}x^{n-2}$.

(b) $\frac{x^2}{(1+x)^3}=x^2-3x^3+6x^4-\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{n(n-1)}{2}x^n$

128. $f(x)=\frac{6}{2-(x-2)}=\frac{6}{4-x} \Rightarrow f'(x)=\frac{6}{(4-x)^2}, f'(1)=\frac{3}{2}$. Then $a_1=\frac{f'(1)}{1!}=f'(1)=\frac{3}{2}$, choice **(C)**.

129. (a) $f(2)\approx T_1(2)=14=4+a_1(2-1) \Rightarrow 10=a_1$.

(b) $a_2=\frac{f''(1)}{2} \Leftrightarrow 11=\frac{f''(1)}{2} \Leftrightarrow 22=f''(1)$. Because $f''(1)=22>0$, the function is concave up at $x=1$ and the tangent line approximation of 14 is less than the actual value of 14.

(c) $T_4(x)=T_3(x)+\frac{f^{(n)}(1)}{4!}(x-1)^4=T_3(x)+\frac{312}{24}(x-1)^4=T_3(x)+13(x-1)^4$

$=4+10(x-1)+11(x-1)^2+15(x-1)^3+13(x-1)^4$

We use this polynomial to approximate $f(2)\approx T_4(2)=53$.

(d) The interval of convergence must be centered at 1, and must have radius $R\geq 3$, since 4 is in the interval. Then $|1-3|=2<3$ so $3\in I$, but $|1-(-2)|=3$, so we cannot tell if $-2$ is in the interval.

(e) $g(x)=\int_1^x f(t)\,dt=\int_1^x\left(4+10(t-1)+11(t-1)^2+15(t-1)^3+13(t-1)^4+\cdots\right)dt$

$=\left[4t+5(t-1)^2+\cdots\right]_1^x=-4+4x-5(x-1)^2+\cdots$.

The second-degree Taylor polynomial for $g$ centered at 1 is $T_{2g}(x)=-4+4x-5(x-1)^2$.

130. Using Table 9.5, $1+(-e)+\frac{(-e)^2}{2!}+\frac{(-e)^3}{3!}+\frac{(-e)^4}{4!}+\cdots=1-e+\frac{e^2}{2!}-\frac{e^3}{3!}+\frac{(e^4}{4!}+\cdots=e^{-e}=\frac{1}{e^e}$, which is choice **(C)**.

131. (a) $T(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\frac{x^8}{4!}+\cdots+\frac{(-1)^n x^{2n}}{n!}+\cdots$. Therefore, $f'(0)=0$, and $\frac{f''(0)}{2!}=-1 \Rightarrow f''(0)=-2$. $f'(0)=0 \Rightarrow f$ has a critical point at $x=0$, and $f''(0)=-2<0 \Rightarrow f$ is concave down at $x=0$. Therefore, $f$ has a local maximum at $x=0$.

(b) $T_6(x)=1-x^2+\frac{x^4}{2}-\frac{x^6}{6}$. $|R_3(1)|=|T_6(1)-f(1)|\le\frac{M}{4!}\cdot|x|^4$, where $|f^{(4)}(x)|\le M$. $f(0)=1$ is a local maximum by part (a), so we can use $M=1$. Therefore $|R_3(1)|\le\frac{1}{24}\cdot|1|^4=0.041\overline{6}<0.1$

(c) Using Table 9.5, $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$, so $e^{-x^2}=1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\frac{x^8}{4!}+\cdots=f(x)$. Then $f'(x)=-2xe^{-x^2}\Rightarrow y'+2xy=(-2x)e^{-x^2}+2x\left(e^{-x^2}\right)=0$, so $y=f(x)=e^{-x^2}$ is a solution to the differential equation $y'+2xy=0$.

132. (a) $\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots \Rightarrow \ln(1+x^3)=x^3-\frac{1}{2}(x^3)^2+\frac{1}{3}(x^3)^3-\frac{1}{4}(x^3)^4+\cdots$

$$=x^3-\tfrac{1}{2}x^6+\tfrac{1}{3}x^9-\tfrac{1}{4}x^{12}+\tfrac{1}{5}x^{15}-\cdots$$

(b) $f'(x)=\frac{3x^2}{1+x^3}=3x^2\left(1-x^3+x^6-x^9+x^{12}-\cdots\right)=3x^2-3x^5+3x^8-3x^{11}+3x^{14}-\cdots$

Then $g(x)=\int_0^x f'(t^2)\,dt=\int_0^x\left(3t^4-3t^{10}+3t^{16}-\cdots\right)dt=\frac{3}{5}x^5-\frac{3}{11}x^{11}+\frac{3}{17}x^{18}-\cdots=\sum_{n=0}^{\infty}\frac{3}{5+6n}x^{5+6n}$

(c) $\frac{3}{5}-\frac{3}{11}=0.3\overline{27}$, and $\frac{3}{17}\approx0.1765<0.2$, so the error in approximating $\int_0^1\frac{3x^4}{1+x^6}\approx0.3\overline{27}$ is less than 0.2.