Graphing

p. 304: 35-49 odd

35. $f(x)=4+\frac{1}{3}x-\frac{1}{2}x^2 \Rightarrow f'(x)=\frac{1}{3}-x$. $f'(x)=0 \Rightarrow x=\frac{1}{3}$. This is the only critical number.

37. $f(x)=2x^3-3x^2-36x \Rightarrow f'(x)=6x^2-6x-36=6(x^2-x-6)=6(x+2)(x-3)$.
$f'(x)=0 \Rightarrow x=-2,3$. These are the only critical numbers.

39. $g(t)=t^4+t^3+t^2+1 \Rightarrow g'(t)=4t^3+3t^2+2t=t(4t^2+3t+2)$. Using the quadratic formula, we see that $4t^2+3t+2$ has no real solutions (its discriminant is negative), so $g'(t)=0$ only if $t=0$. Hence, the only critical number is 0.

41. $g(y)=\dfrac{y-1}{y^2-y+1} \Rightarrow$

$$g'(y)=\frac{(y^2-y+1)(1)-(y-1)(2y-1)}{(y^2-y+1)^2}=\frac{y^2-y+1-(2y^2-3y+1)}{(y^2-y+1)^2}=\frac{-y^2+2y}{(y^2-y+1)^2}=\frac{y(2-y)}{(y^2-y+1)^2}.$$

$g'(y)=0 \Rightarrow y=0,2$. The expression y^2-y+1 is never equal to 0, so $g'(y)$ exists for all real numbers. The critical numbers are 0 and 2.

43. $h(t)=t^{3/4}-2t^{1/4} \Rightarrow h'(t)=\frac{3}{4}t^{-1/4}-\frac{2}{4}t^{-3/4}=\frac{1}{4}t^{-3/4}(3t^2-2)=\dfrac{3\sqrt{t}-2}{4\sqrt[4]{t^3}}$.

$h'(t)=0 \Rightarrow 3\sqrt{t}=2 \Rightarrow \sqrt{t}=\frac{2}{3} \Rightarrow t=\frac{4}{9}$. $h'(0)$ does not exist, so the critical numbers are 0 and $\frac{4}{9}$.

45. $F(x)=x^{4/5}(x-4)^2 \Rightarrow$

$$F'(x)=x^{4/5}2(x-4)+(x-4)^2\cdot\tfrac{4}{5}x^{-1/5}=\tfrac{1}{5}x^{-1/5}(x-4)[5\cdot x\cdot 2+(x-4)4]=\frac{(x-4)(14x-16)}{5x^{1/5}}$$

$=\dfrac{2(x-4)(7x-8)}{5x^{1/5}}$. $F'(x)=0 \Rightarrow x=4,\frac{8}{7}$. $F'(0)$ does not exist.

Thus, the three critical numbers are $0, \frac{8}{7}$, and 4.

47. $f(\theta)=2\cos\theta+\sin^2\theta \Rightarrow f'(\theta)=-2\sin\theta+2\sin\theta\cos\theta$. $f'(\theta)=0 \Rightarrow 2\sin\theta(\cos\theta-1)=0 \Rightarrow$ $\sin\theta=0$ or $\cos\theta=1 \Rightarrow \theta=n\pi$ [n an integer] or $\theta=2n\pi$. The solutions $\theta=n\pi$ include the solutions $\theta=2n\pi$, so the critical numbers are $\theta=n\pi$.

49. $f(x)=x^2e^{-3x} \Rightarrow f'(x)=x^2(-3e^{-3x})+e^{-3x}(2x)=xe^{-3x}(-3x+2)$. $f'(x)=0 \Rightarrow x=0,\frac{2}{3}$ [e^{-3x} is never equal to 0]. $f'(x)$ always exists, so the critical numbers are 0 and $\frac{2}{3}$.

p. 331: 11-12, 14-15, 17-25 odd, 26-29, 45-53

11. (a) Since $f'(x)>0$ on $(1,5)$ f is increasing on this interval. Since $f'(x)<0$ on $(0,1)$ and $(5,6)$ f is decreasing on these intervals.
(b) Since $f'(x)=0$ at $x=1$ and f' changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x=1$. Since $f'(x)=0$ at $x=5$ and f' changes from positive to negative there, f changes from increasing to decreasing and has a local maximum at $x=5$.

12. (a) $f'(x)=0$ and f is increasing on $(0,1)$ and $(5,7)$. $f'(x)<0$ and f is decreasing on $(1,5)$ and $(7,8)$.

(b) Since $f'(x)=0$ at $x=1$ and $x=7$ and f' changes from positive to negative at both values, f changes from increasing to decreasing and has local maxima at $x=1$ and $x=7$. Since $f'(x)=0$ at $x=5$ and f' changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x=5$.

14. (a) f is increasing when f' is positive. This happens on the intervals $(0,4)$ and $(6,8)$.

(b) f has a local maximum where it changes from increasing to decreasing, that is, where f' changes from positive to negative (at $x=4$ and $x=8$). Similarly, f has a local minimum where f' changes from negative to positive (at $x=6$).

(c) f is concave up where f' is increasing (hence f'' is positive). This happens on $(0,1),(2,3)$ and $(5,7)$. Similarly, f is concave down where f' is decreasing, that is, on $(1,2),(3,5)$ and $(7,9)$.

(d) f has an inflection point where the concavity changes. This happens at $x=1,2,3,5$ and 7.

15. $f'(x)<0 \Rightarrow f$ is decreasing and $f''(x)>0 \Rightarrow f$ is concave up. The graph that depicts a decreasing, concave up segment is (**B**).

17. (a) $f(x)=2x^3-9x^2 12x-3 \Rightarrow f'(x)=6x^2-18x+12=6(x^2-3x+2)=6(x-1)(x-2)$.

Interval	$x+1$	$x-3$	$f'(x)$	f
$x<-1$	–	–	+	increasing on $(-\infty,1)$
$1<x<2$	+	–	–	decreasing on $(1,2)$
$x>2$	+	+	+	increasing on $(2,\infty)$

(b) f changes from increasing to decreasing at $x=1$ and from decreasing to increasing at $x=2$. Thus, $f(1)=2$ is a local maximum value and $f(2)=1$ is a local minimum value.

(c) $f''(x)=12x-18=12(x-\frac{3}{2})$. $f''(x)>0 \Leftrightarrow x>\frac{3}{2}$ and $f''(x)<0 \Leftrightarrow x<\frac{3}{2}$. Thus, f is concave up on $(\frac{3}{2},\infty)$ and concave down on $(-\infty,\frac{3}{2})$. There is an inflection point at $(\frac{3}{2},\frac{3}{2})$.

19. (a) $f(x)=\dfrac{x}{x^2+1} \Rightarrow f'(x)=\dfrac{(x^2+1)\cdot 1-x(2x)}{(x^2+1)^2}=\dfrac{1-x^2}{(x^2+1)^2}=-\dfrac{(x+1)(x-1)}{(x^2+1)^2}$. Thus, $f'(x)>0$ if $(x+1)(x-1)<0 \Leftrightarrow -1<x<1$, and $f'(x)<0$ if $x<-1$ or $x>1$. So f is increasing on $(-1,1)$ and f is decreasing on $(-\infty,-1)$ and $(1,\infty)$.

(b) f changes from decreasing to increasing at $x=-1$ and from increasing to decreasing at $x=1$. Thus, $f(-1)=-\frac{1}{2}$ is a local minimum value and $f(1)=\frac{1}{2}$ is a local maximum value.

(c) $f''(x)=\dfrac{(x+1)^2(-2x)-(1-x^2)[(2(x^2+1)(2x)]}{(x+1)^4}=\dfrac{(x+1)^2(-2x)[(x^2+1)+2(1-x^2)]}{(x+1)^4}=\dfrac{2x(x^2-3)}{(x+1)^3}$.

$f''(x)>0 \Leftrightarrow -\sqrt{3}<x<0$ or $x>\sqrt{3}$, and $f''(x)<0 \Leftrightarrow x<-\sqrt{3}$ or $0<x<\sqrt{3}$. Thus, f is concave up on $(-\sqrt{3},0)$ and $(\sqrt{3},\infty)$ and concave down on $(-\infty,-\sqrt{3})$ and $(0,\sqrt{3})$. There are inflection points at $(-\sqrt{3},-\sqrt{3}/4)$, $(0,0)$, and $(\sqrt{3},\sqrt{3}/4)$.

21. (a) $f(x)=\cos^2 x-2\sin x,\ 0\le x\le 2\pi$. $f'(x)=-2\cos x\sin x-2\cos x=-2\cos x(1+\sin x)$. Note that $1+\sin x\ge 0$, with equality $\Leftrightarrow \sin x=-1 \Leftrightarrow x=\frac{3\pi}{2}$ $[\text{since } 0\le x\le 2\pi] \Rightarrow \cos x=0$. Thus, $f'(x)>0 \Leftrightarrow \cos x<0 \Leftrightarrow \frac{\pi}{2}<x<\frac{3\pi}{2}$ and $f'(x)<0 \Leftrightarrow \cos x>0 \Leftrightarrow 0<x<\frac{\pi}{2}$ or $\frac{3\pi}{2}<x<2\pi$. Thus, f is increasing on $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ and f is decreasing on $\left(0,\frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2},2\pi\right)$.

(b) f changes from decreasing to increasing at $x=\frac{\pi}{2}$ and from increasing to decreasing at $x=\frac{3\pi}{2}$. Thus, $f\left(\frac{\pi}{2}\right)=-2$ is a local minimum value and $f\left(\frac{3\pi}{2}\right)=2$ is a local maximum value.

(c) $f''(x)=2\sin x(1+\sin x)-2\cos^2 x=2\sin x+2\sin^2 x-2(1-\sin^2 x)$

$$=4\sin^2 x+2\sin x-2=2(2\sin x-1)(\sin x+1)$$

so $f''(x)>0 \Leftrightarrow \sin x>\frac{1}{2} \Leftrightarrow \frac{\pi}{6}<x<\frac{5\pi}{6}$, and $f''(x)<0 \Leftrightarrow \sin x<\frac{1}{2}$ and $\sin x\ne -1 \Leftrightarrow 0<x<\frac{\pi}{6}$ or $\frac{5\pi}{6}<x<\frac{3\pi}{2}$ or $\frac{3\pi}{2}<x<2\pi$. Thus, f is concave up on $\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$ and concave down on $\left(0,\frac{\pi}{6}\right),\left(\frac{5\pi}{6},\frac{3\pi}{2}\right)$, and $\left(\frac{3\pi}{2},2\pi\right)$. There are inflection points at $\left(\frac{\pi}{6},-\frac{1}{4}\right)$ and $\left(\frac{5\pi}{6},-\frac{1}{4}\right)$.

23. (a) $f(x)=x^2\ln x \Rightarrow f'(x)=x^2(1/x)+(\ln x)(2x)=x+2x\ln x=x(1+2\ln x)$. The domain of f is $(0,\infty)$, so the sign of f' is determined solely by the factor $1+2\ln x$. $f'(x)>0 \Leftrightarrow \ln x>-\frac{1}{2}\ x>e^{-1/2}$ and $f'(x)<0 \Leftrightarrow 0<x<e^{-1/2}$. So f is increasing on $\left(e^{-1/2},\infty\right)$ and f is decreasing on $\left(0,e^{-1/2}\right)$.

(b) f changes from decreasing to increasing at $x=e^{-1/2}$.
Thus, $f\left(e^{-1/2}\right)=\left(e^{-1/2}\right)^2\ln\left(e^{-1/2}\right)=e^{-1}\left(-\frac{1}{2}\right)=-\frac{1}{2e}$ is a local minimum value.

(c) $f'(x)=x(1+2\ln x) \Rightarrow f''(x)=x(2/x)+(1+2\ln x)\cdot 1=2+1+2\ln=3+2\ln x$. $f''(x)>0 \Leftrightarrow 3+2\ln x>0 \Leftrightarrow \ln x>-\frac{3}{2} \Leftrightarrow x>e^{-3/2}$. Thus, f is concave up on $\left(e^{-3/2},\infty\right)$ and f is concave down on $\left(0,e^{-3/2}\right)$. There is a point of inflection at $\left(e^{-3/2},f\left(e^{-3/2}\right)\right)=\left(e^{-3/2},-3/2e^3\right)$.

25. (a) $f(x)=x^4e^{-x} \Rightarrow f'(x)=x^4\left(-e^{-x}\right)+e^{-x}(4x^3)=x^3e^{-x}(-x+4)$. Thus, $f'(x)>0$ if $0<x<4$ and $f'(x)<0$ if $x<0$ or $x>4$. So f is increasing on $(0,4)$ and f is decreasing on $(-\infty,0)$ and $(4,\infty)$.

(b) f changes from decreasing to increasing at $x=0$ and from increasing to decreasing at $x=4$. Thus, $f(0)=0$ is a local minimum value and $f(4)=256/e^4$ is a local maximum value.

(c) $f'(x)=e^{-x}(-x^4+4x^3) \Rightarrow$

$$f''(x)=e^{-x}(-4x^3+12x^2)+(-x^4+4x^3)\left(-e^{-x}\right)=e^{-x}[(-4x^3+12x^2)-(-x^4+4x^3)]$$

$$=e^{-x}(x^4-8x^3+12x^2)=x^2e^{-x}(x^2-8x+12)=x^2e^{-x}(x-2)(x-6)$$

$f''(x)>0 \Leftrightarrow x<2$ [excluding 0] or $x>6$ and $f''(x)<0 \Leftrightarrow 2<x<6$. Thus, f is concave upward on $(-\infty,2)$ and $(6,\infty)$ and f is concave down on $(2,6)$. There are inflection points at $\left(2,16e^{-2}\right)$ and $\left(6,1296e^{-6}\right)$.

26. $f(x)=2x^3+x \Rightarrow f'(x)=6x^2+1 \Rightarrow f'(x)>0$ for all x so f is increasing for all values of x (III), but f has no local extrema. $f''(x)=12x=0 \Leftrightarrow x=0$. $f''(x)>0 \Rightarrow x>0$ and $f''(x)<0 \Rightarrow x<0$. So, f has an infection point at $(0,0)$ (II). Therefore, the correct choice is (**D**), II and III.

27. $h(x)=x^3+2x^2-4x+1 \Rightarrow h'(x)=3x^2+4x-4=(x+2)(3x-2)$. $h'(x)=0 \Leftrightarrow x=-2$, or $\frac{2}{3}$. $h'(x)<0 \Leftrightarrow -2<x<\frac{2}{3} \Rightarrow h$ is decreasing on $\left[-2,\frac{2}{3}\right]$. This is option (**A**).

28. $f''(x)=x(x-2)^2(x+1)^3=0 \Leftrightarrow x=-1,0,2.$ Using a sign chart for $f''(x),$ we have

Interval	$f''(x)=x(x-2)^2(x+1)^3$	Concavity
$(-\infty,-1)$	$f''(-2)=32>0$	up
$(-1,0)$	$f''\left(-\frac{1}{2}\right)=-\frac{25}{64}<0$	down
$(0,2)$	$f''(1)=8>0$	up
$(2,\infty)$	$f''(3)=192>0$	up

Therefore, f has inflection points at $x=-1$ and $x=0,$ which is option (**D**).

29. $f(x)=xe^x-e^x \Rightarrow f'(x)=xe^x+e^x\cdot 1-e^x=xe^x.$ $f'(x)=0 \Leftrightarrow x=0.$
$f'(x)>0 \Leftrightarrow x>0$ and $f'(x)<0 \Leftrightarrow x<0.$ So f has a relative maximum at $x=0$ and no relative minima. $f''(x)=xe^x+e^x=e^x(x+1).$ $f''(x)=0 \Leftrightarrow x=-1, f''(x)>0 \Leftrightarrow x>-1$ and $f''(x)<0 \Leftrightarrow x<-1.$ Therefore, f has an inflection point at $x=-1.$ The correct choice is (**B**), one relative maximum and one point of inflection.

45. (a) $\frac{dy}{dx}>0$ (f is increasing) and $\frac{d^2y}{dx^2}>0$ (f is concave up) at point B.

(b) $\frac{dy}{dx}<0$ (f is decreasing) and $\frac{d^2y}{dx^2}<0$ (f is concave down) at point E.

(c) $\frac{dy}{dx}<0$ (f is decreasing) and $\frac{d^2y}{dx^2}>0$ (f is concave up) at point A.

Note: At C, $\frac{dy}{dx}>0$ and $\frac{d^2y}{dx^2}<0.$ At D, $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}\le 0.$

46. $f(x)=\frac{1}{x\ln x}=(x\ln x)^{-1} \Rightarrow f'(x)=-(x\ln x)^{-2}\left[x\cdot\frac{1}{x}+\ln x\cdot 1\right]=-\frac{1+\ln x}{(x\ln x)^2}.$
$f'(x)=0 \Leftrightarrow 1+\ln x=0 \Leftrightarrow \ln x=-1 \Leftrightarrow e^{\ln x}=x=e^{-1}=\frac{1}{e}.$ Note that f is undefined for $x\le 0$ $x=1.$ So $f'(x)<0 \Leftrightarrow \frac{1}{e}<x<1$ and $x>1,$ and $f'(x)>0 \Leftrightarrow 0<x<\frac{1}{e}.$ Therefore, the local maximum value occurs at $x=\frac{1}{e},$ and the local maximum value is $f\left(\frac{1}{e}\right)=-e,$ which is choice (**B**).

47. $f(x)=x^3-2x^2-4x+5 \Rightarrow f'(x)=3x^2-4x-4=(3x+2)(x-2).$ $f'(x)=0 \Leftrightarrow x=2,-\frac{2}{3}.$
$f'(x)<0 \Leftrightarrow -\frac{2}{3}<x<2$ and $f'(x)>0 \Leftrightarrow x<-\frac{2}{3}x$ or $x>2.$ Therefore, f has a local minimum at $x=2,$ and a local maximum at $x=-\frac{2}{3}.$ $f''(x)=6x-4=0 \Leftrightarrow x=\frac{2}{3}.$ So, f does not have an inflection point at $x=-\frac{2}{3}.$ The correct choice is (**D**), the function is decreasing on $\left[-\frac{2}{3},2\right].$

48. $f(x)=\sin x\cdot e^{\cos^2 x} \Rightarrow f'(x)=\sin xe^{\cos^2 x}(2\cos x(-\sin x))+e^{\cos^2 x}(\cos x)=e^{\cos^2 x}\cos x(1-2\sin^2 x).$

$f'(x)=0 \Leftrightarrow \cos x(1-2\sin^2 x)=0 \Leftrightarrow \cos x=2\sin^2 x \Leftrightarrow 1=2\frac{\sin^2 x}{\cos x} \Leftrightarrow \frac{1}{2}=\sin x\tan x$

$\Leftrightarrow x=\frac{\pi}{4}+\frac{n\pi}{4},$ n an integer. So among the given choices, the correct choice is (**A**) $\frac{\pi}{2}.$

49. On the interval $0<x<4,$ $f'(x)=\frac{\sin\left(\frac{\pi x}{2}\right)}{x}=0 \Leftrightarrow \sin\left(\frac{\pi x}{2}\right)=0 \Leftrightarrow x=2.$ Note that $f(0)$ is undefined but 0 is not in the given interval. $0<x<2 \Rightarrow f'(x)>0$ and $2<x<4 \Rightarrow f'(x)<0.$ Therefore, f has a local maximum at $x=2,$ and no local minima in the interval. The correct choice is (**A**).

50. $f(x) = xe^{-x^2} \Rightarrow f'(x) = xe^{-x^2}(-2x) + e^{-x^2} \cdot 1 = e^{-x^2}(1-2x^2)$.

$f'(x) = 0 \Leftrightarrow 1-2x^2 = 0 \Leftrightarrow \frac{1}{2} = x^2 \Leftrightarrow \pm\frac{1}{\sqrt{2}} = x$. $f'(x) > 0 \Leftrightarrow -\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2}$ and $f'(x) < 0 \Leftrightarrow x < -\frac{\sqrt{2}}{2}$ and $x > \frac{\sqrt{2}}{2}$. Thus f has a local minimum at $x = -\frac{\sqrt{2}}{2}$ and a local maximum at $x = \frac{\sqrt{2}}{2}$. Therefore, the function is decreasing on $\left(-\infty, -\frac{\sqrt{2}}{2}\right]$ and $\left[\frac{\sqrt{2}}{2}, \infty\right)$.

$f''(x) = e^{-x^2}(-4x) + (1-2x^2)e^{-x^2}(-2x) = e^{-x^2}(-4x-2x+4x^3) = e^{-x^2}(4x^3-6x) = 2xe^{-x^2}(2x^2-3)$.
$f''(x) = 0 \Leftrightarrow x = 0$ or $2x^2 - 3 = 0 \Leftrightarrow x^2 = \frac{3}{2} \Leftrightarrow x = \pm\sqrt{\frac{3}{2}}$. $f''(x) < 0 \Leftrightarrow x < -\sqrt{\frac{3}{2}}$ and $0 < x < \sqrt{\frac{3}{2}}$.
Thus f is not concave down over its entire domain. The true statement is (**D**).

51. $y = \dfrac{1}{x} + \ln x \Rightarrow y' = -\dfrac{1}{x^2} + \dfrac{1}{x} \Rightarrow y'' = \dfrac{2}{x^3} - \dfrac{1}{x^2} = \dfrac{2-x}{x^3}$. $y'' = 0 \Leftrightarrow x = 2$, $y'' > 0 \Leftrightarrow x > 2$ and $y'' < 0 \Leftrightarrow x < 2$. Therefore, y has a point of inflection at (**A**) $x = 2$.

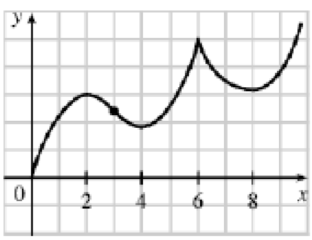
52. (a) f is increasing where f' is positive, that is on $(0,2), (4,6)$ and $(8,\infty)$; and decreasing where f' is negative, that is, on $(2,4)$ and $(6,8)$.

(b) f has local maxima where f' changes from positive to negative, at $x = 2$ and at $x = 6$, and local minima where f' changes from negative to positive, at $x = 4$ and at $x = 8$.

(c) f is concave upward (CU) where f' is increasing, that is, on $(3,6)$ and $(6,\infty)$, and concave downward (CD) where f' is decreasing, that is, on $(0,3)$.

(d) There is a point of inflection where f changes from being CD to being CU, that is, at $x = 3$.

(e)



53. (a) f is increasing where f' is positive, on $(1,6)$ and $(8,\infty)$, and decreasing where f' is negative, on $(0,1)$ and $(6,8)$.

(b) f has a local maximum where f' changes from positive to negative, at $x = 6$, and local minima where f' changes from negative to positive, at $x = 1$ and at $x = 8$.

(c) f is concave upward where f' is increasing, that is, on $(0,2), (3,5)$, and $(7,\infty)$, and concave downward where f' is decreasing, on $(2,3)$ and $(5,7)$.

(d) There are points of inflection where f changes its direction of concavity, at $x = 2, x = 3, x = 5,$ and $x = 7$.